Magnetohydrodynamics for Pedestrians

DESY physics seminar

Zeuthen 23.11.2016

Rolf Bühler

What is Plasma?

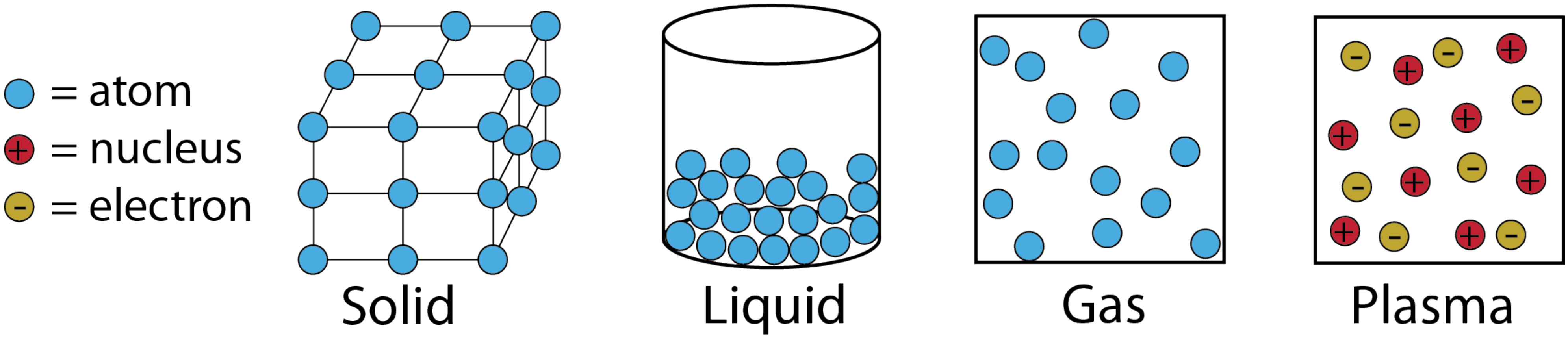
e.g. Hydrogen:
(P = 1 atm)

T < 14 K

14 K < T < 20 K

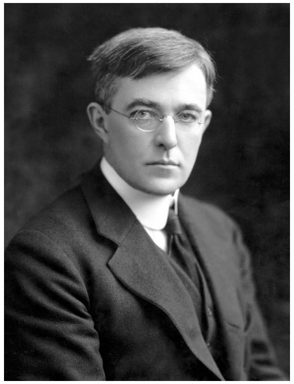
20 K < T ≈< 15000 K

T >≈ 15000 K



$$k_B \times T \approx 13.6 \text{ eV} \rightarrow T \approx 13.6 / 8.6\times10^{-5} \text{ K}^{-1} \approx 15\,000 \text{ K}$$

Below $k_B \times T \approx 4.52$ eV (T≈5000 K) H is bounded into H_2 molecule



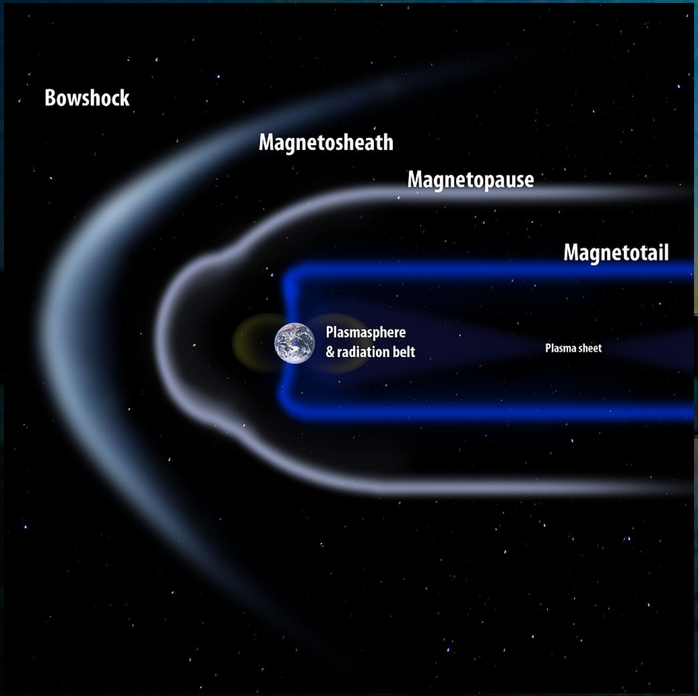
Word “**plasma**” attributed to Nobel prize Chemist Irvin Langmuir, who was reminded of corpuscles being carried in the blood

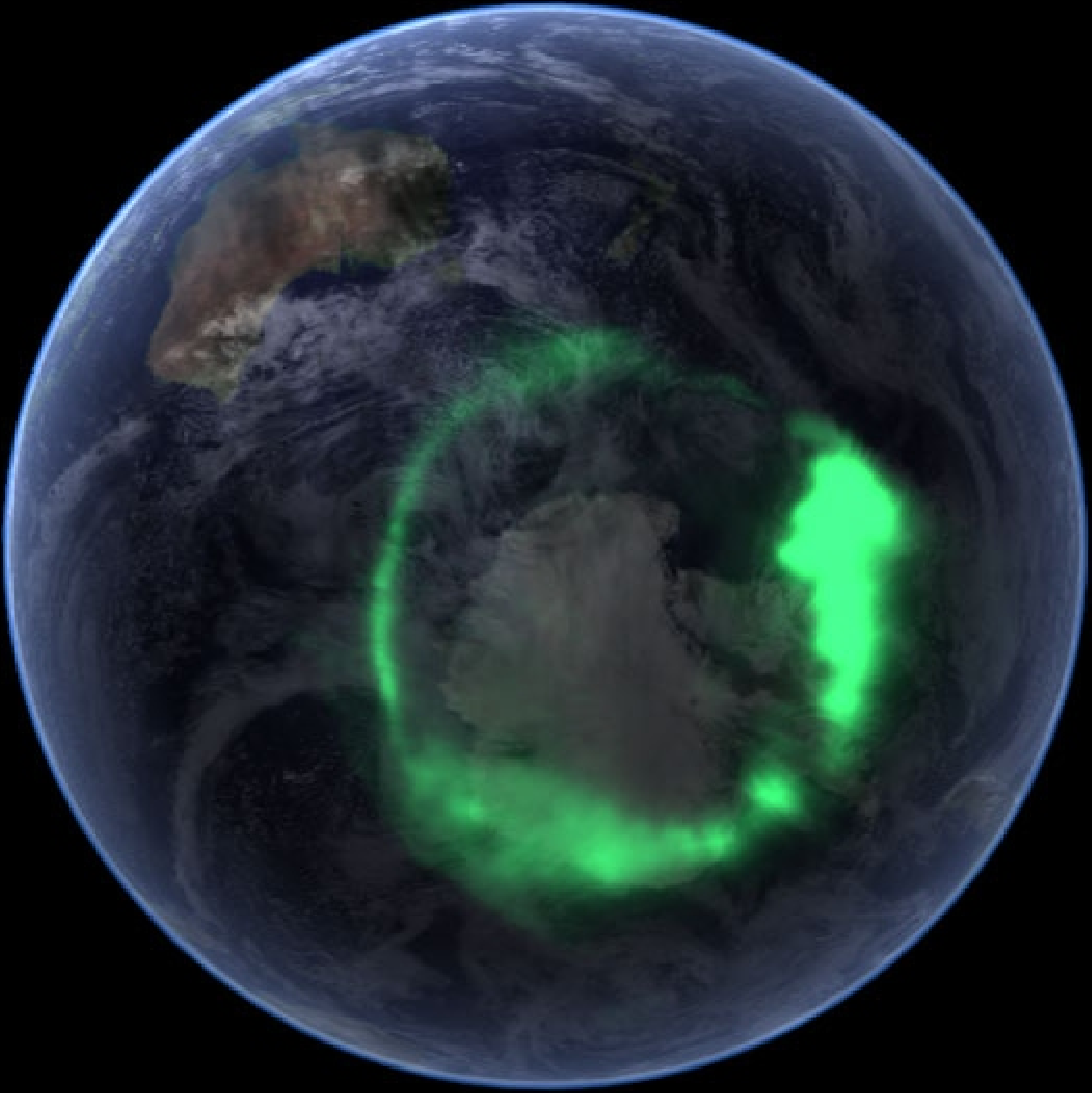
Plasma on Earth

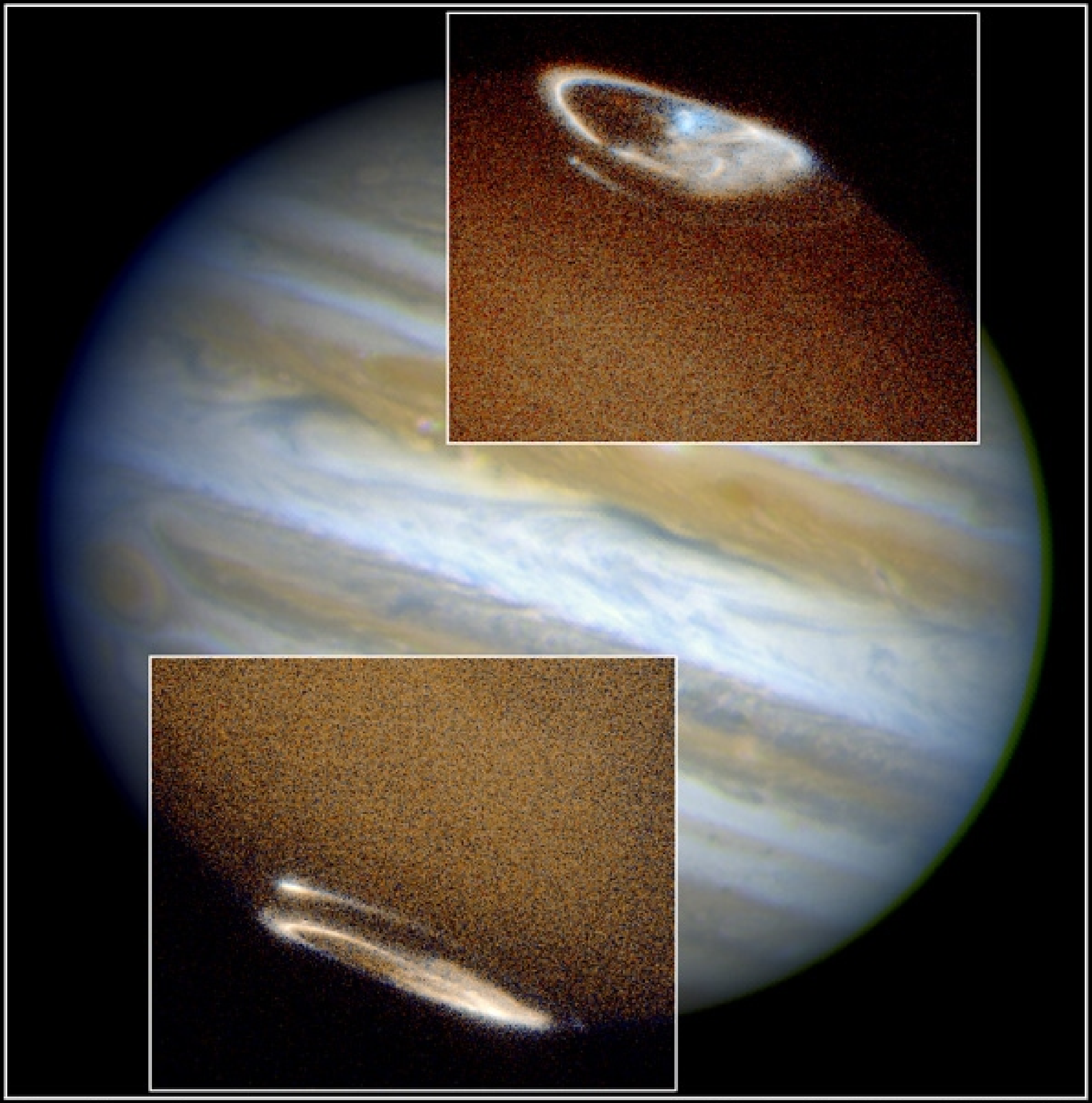
Plasma on Earth

..but also..

Plasmas on-top of Earth

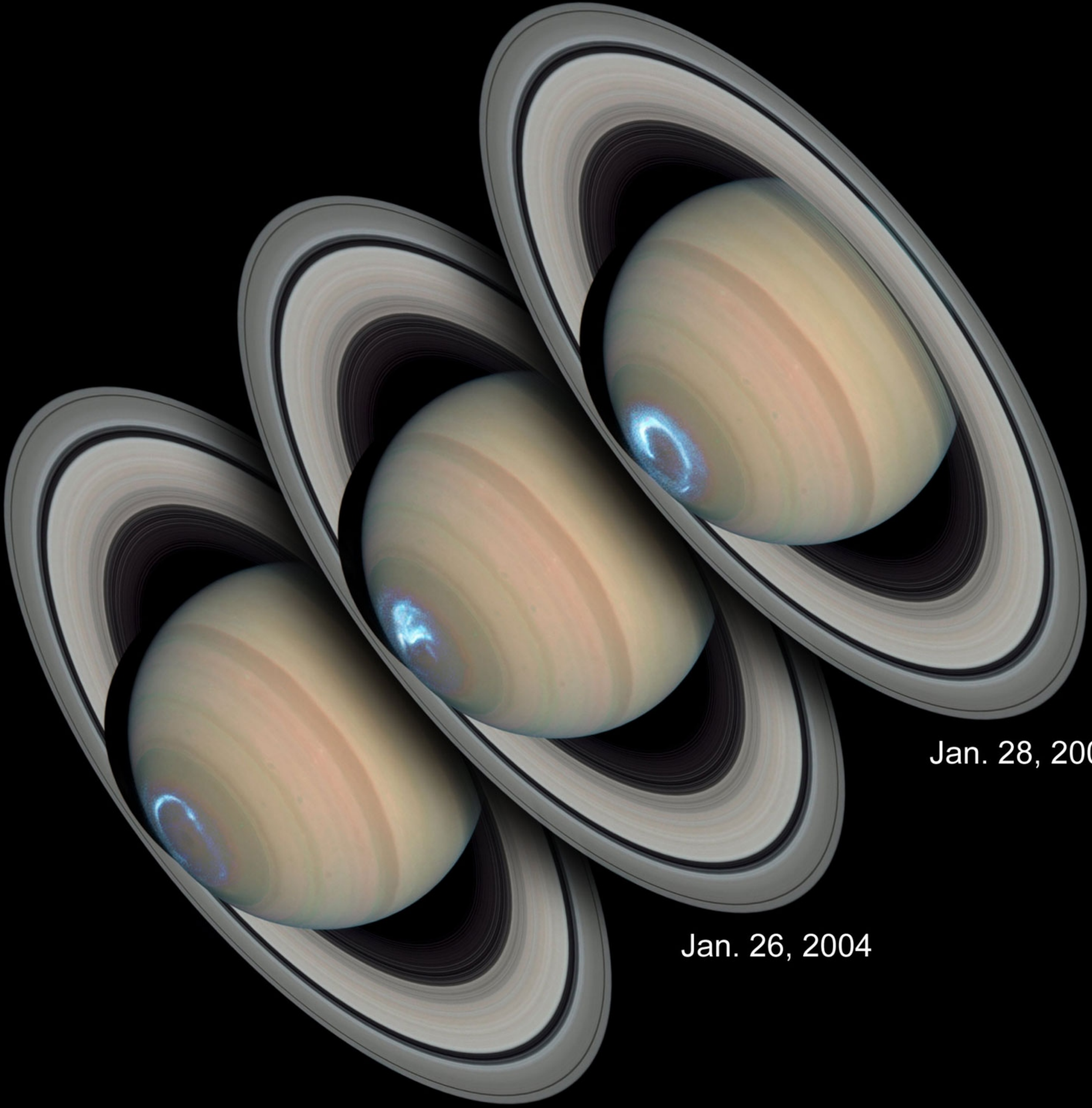






Jupiter Aurora HST • STIS • WFPC2

PRC98-04 • ST ScI OPO • January 7, 1998
J. Clarke (University of Michigan) and NASA

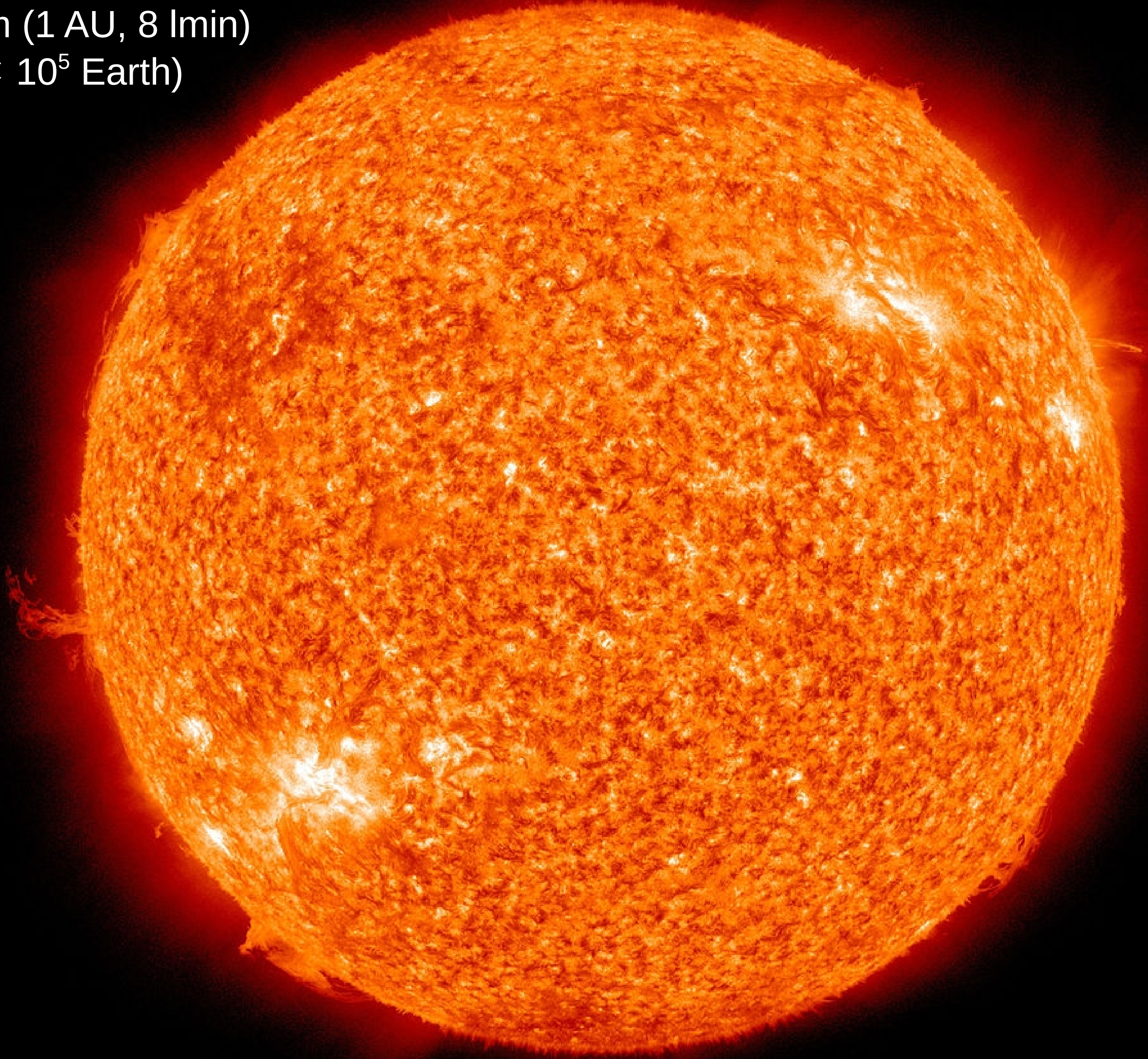


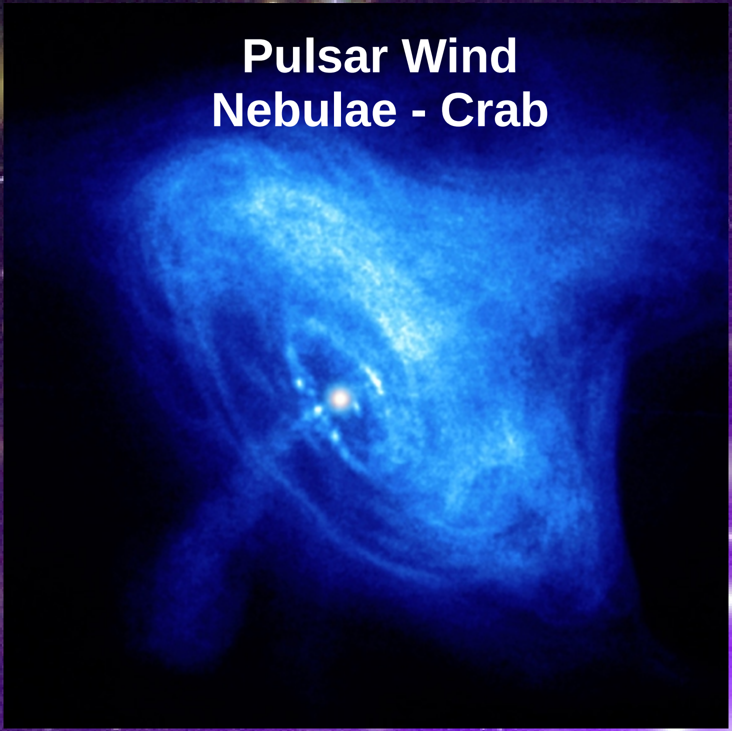
Jan. 28, 2004

Jan. 26, 2004

Jan. 24, 2004

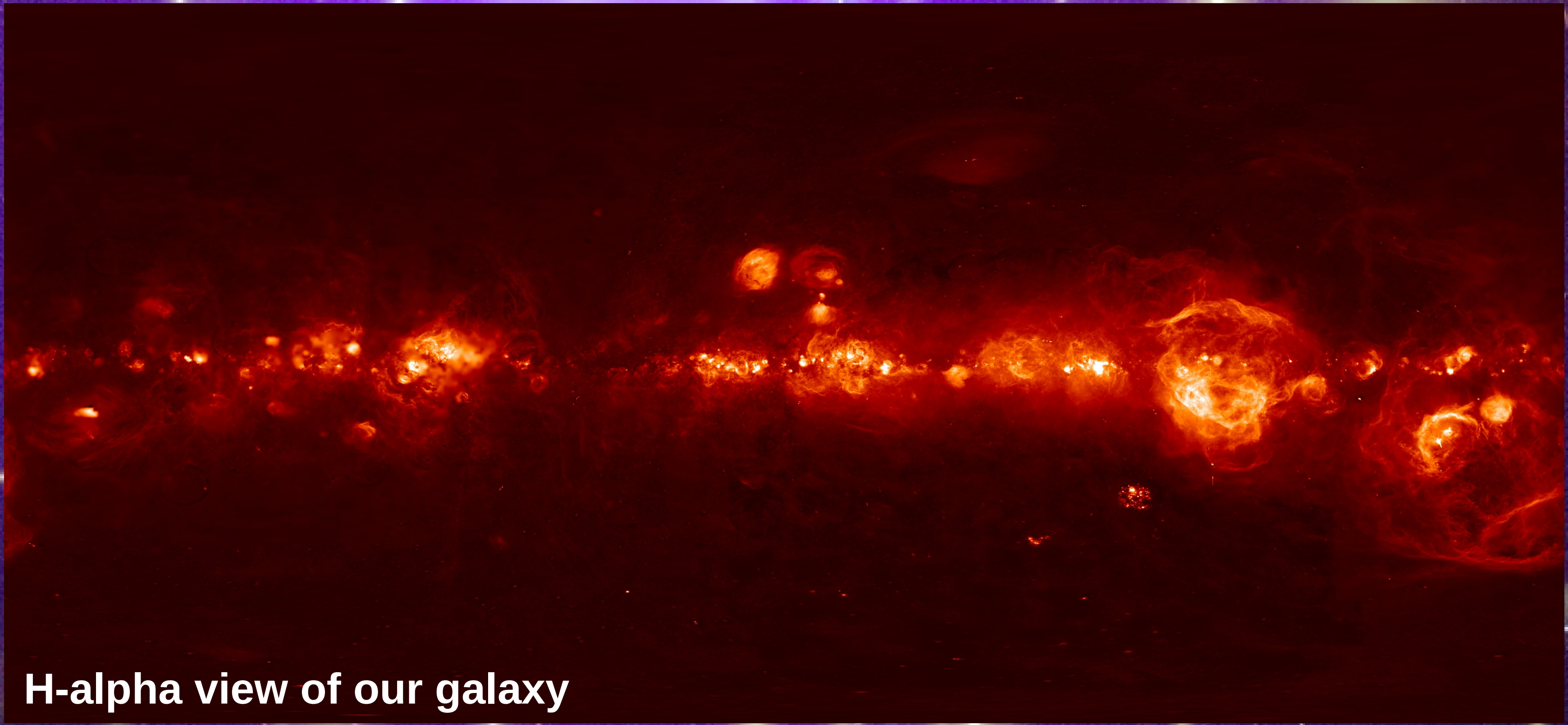
Distance: 1.5 × 10^8 km (1 AU, 8 lmin)
Mass: 2 × 10^{30} kg (3 × 10^5 Earth)




Pulsar Wind Nebulae - Crab

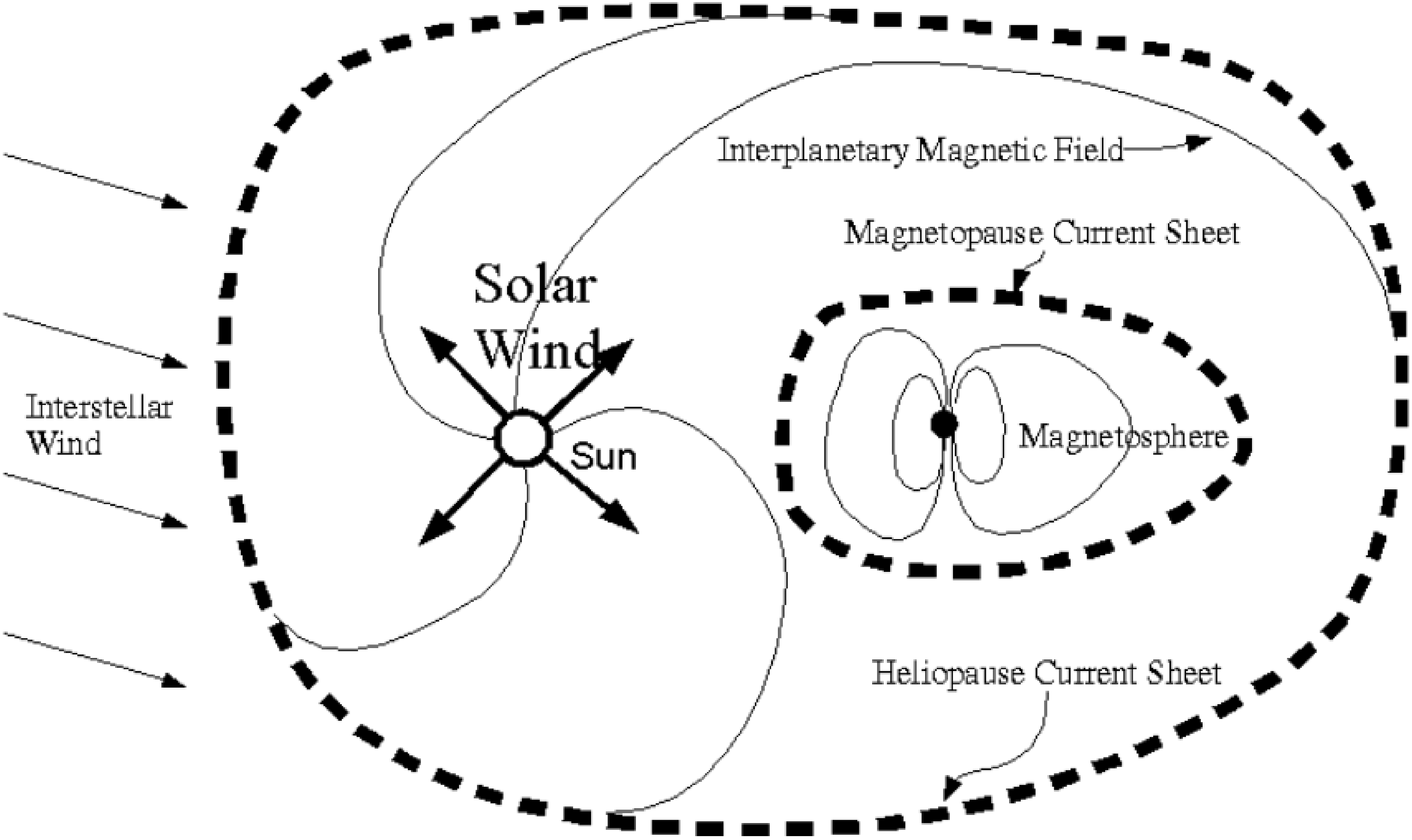

Supernovae Remnants Tycho


Active Galactic Nuclei - M87


H-alpha view of our galaxy

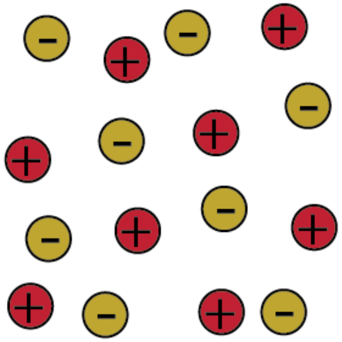
Galaxy Cluster Abell 1689

Universe of Plasma Bubbles



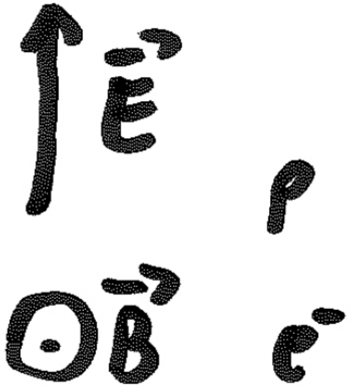
Plasma Microphysics

Fluctuations at the plasma frequency due to thermal energy at Debye length:



$$\omega_{pl} = \sqrt{\frac{n_0 e^2}{m_e \epsilon_0}} \qquad \lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{e^2 n_0}}$$

Often surprising effects, for example currents due to drifts:



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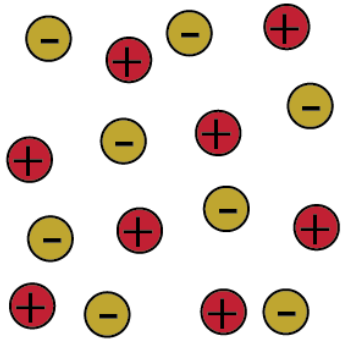

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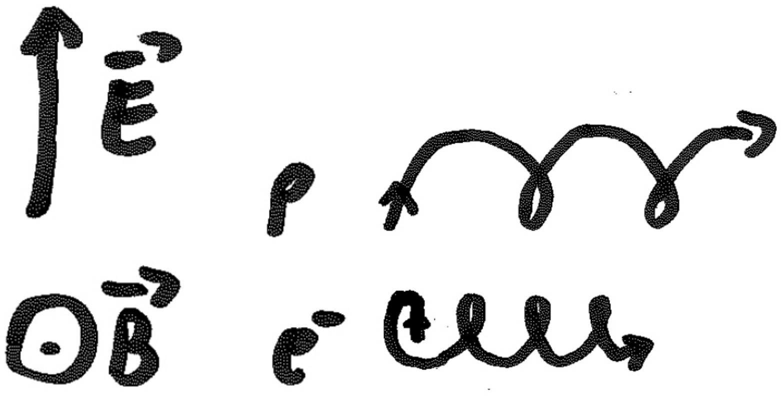
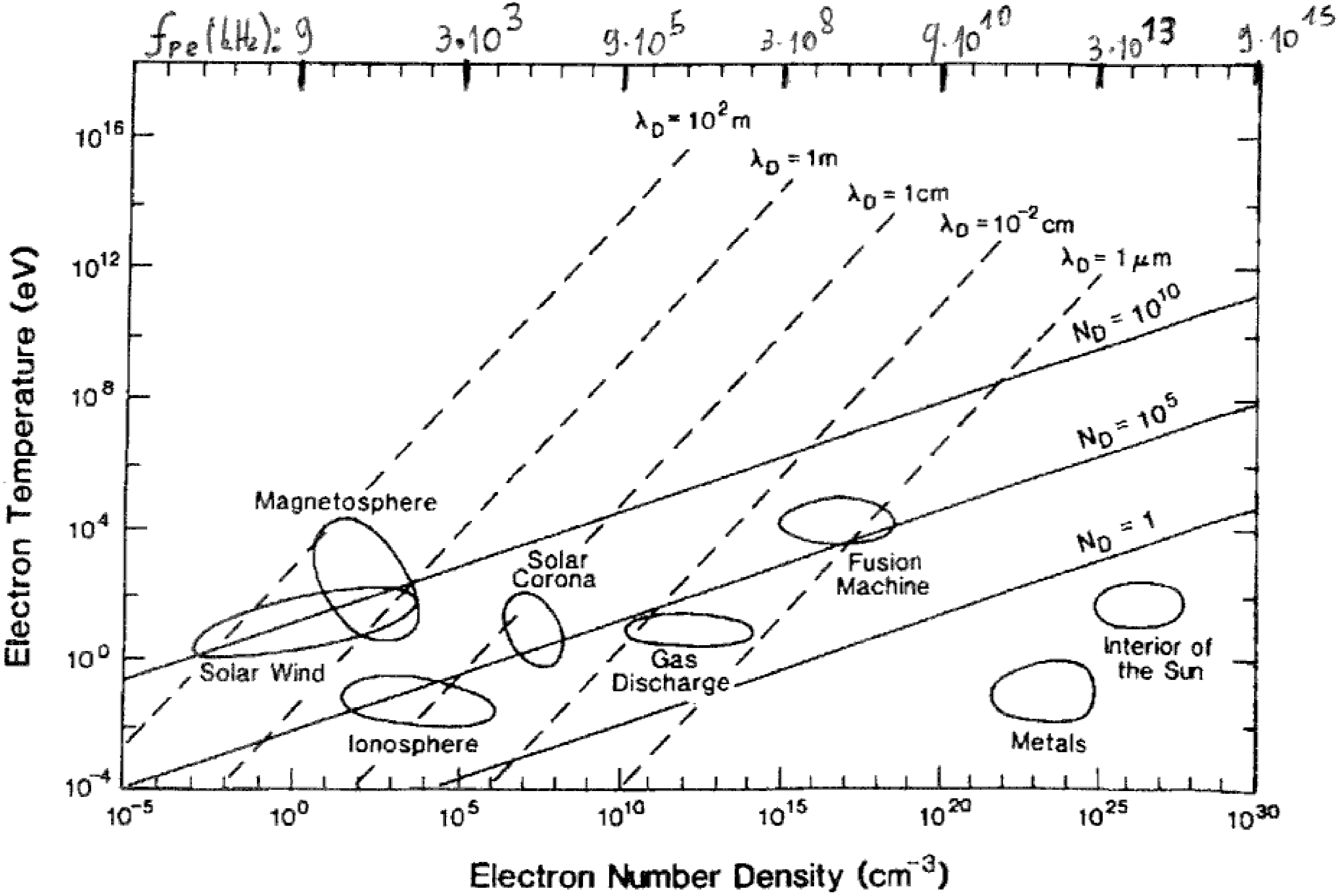




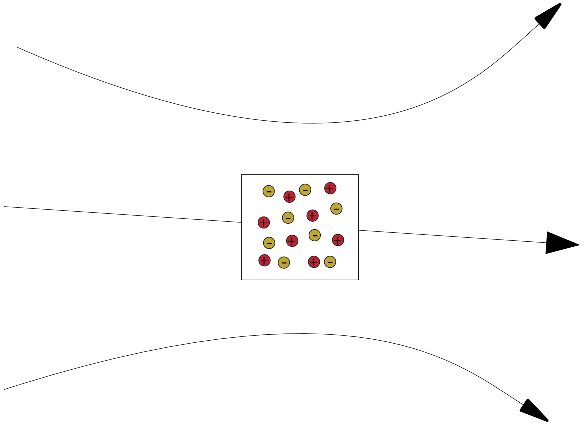
fpe (kHz): 9
3·10^3
9·10^5
3·10^8
9·10^10
3·10^13
9·10^15
λD = 10^2 m
λD = 1m
λD = 1cm
λD = 10^-2 cm
λD = 1 μm
ND = 10^10
ND = 10^5
ND = 1
Magnetosphere
Solar Wind
Solar Corona
Ionosphere
Gas Discharge
Fusion Machine
Interior of the Sun
Metals
Electron Temperature (eV)
10^16
10^12
10^8
10^4
10^0
10^-4
10^-5
10^0
10^5
10^10
10^15
10^20
10^25
10^30
Electron Number Density (cm^-3)

Plasma Descriptions

1) Exact: Calculate positions, velocities and electromagnetic fields for N particles. In interstellar space $n \approx 1\ \mathrm{cm}^{-3}$, so $N=10^{15}$ in 1 km^3 volume. Typically unfeasible.

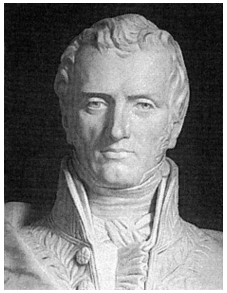
2) Distribution function: Calculate evolution of distribution function $f(x_i,v_j)\ dx^3\ dv^3$. Results in Vlasov equation. Precise but still often unfeasible.

3) Magnetohydrodynamics (MHD): Use equations of state and apply fluid dynamics with Maxwell's equations. Not precise, but often a good approximation.



- Density - ρ
- Pressure - P
- Temperature – T
- Velocity - **v**
- Electric Field - **E**
- Magnetic Field - **B**

Hydrodynamics



Navier-Stokes equation (momentum conservation)

$$(\partial_t + \vec{v}\nabla)\rho\vec{v} = \rho\vec{g} - \nabla P + ...$$

Mass conservation

$$\partial_t\rho = -\nabla(\rho\vec{v})$$

Adiabatic equation of state (energy conservation)

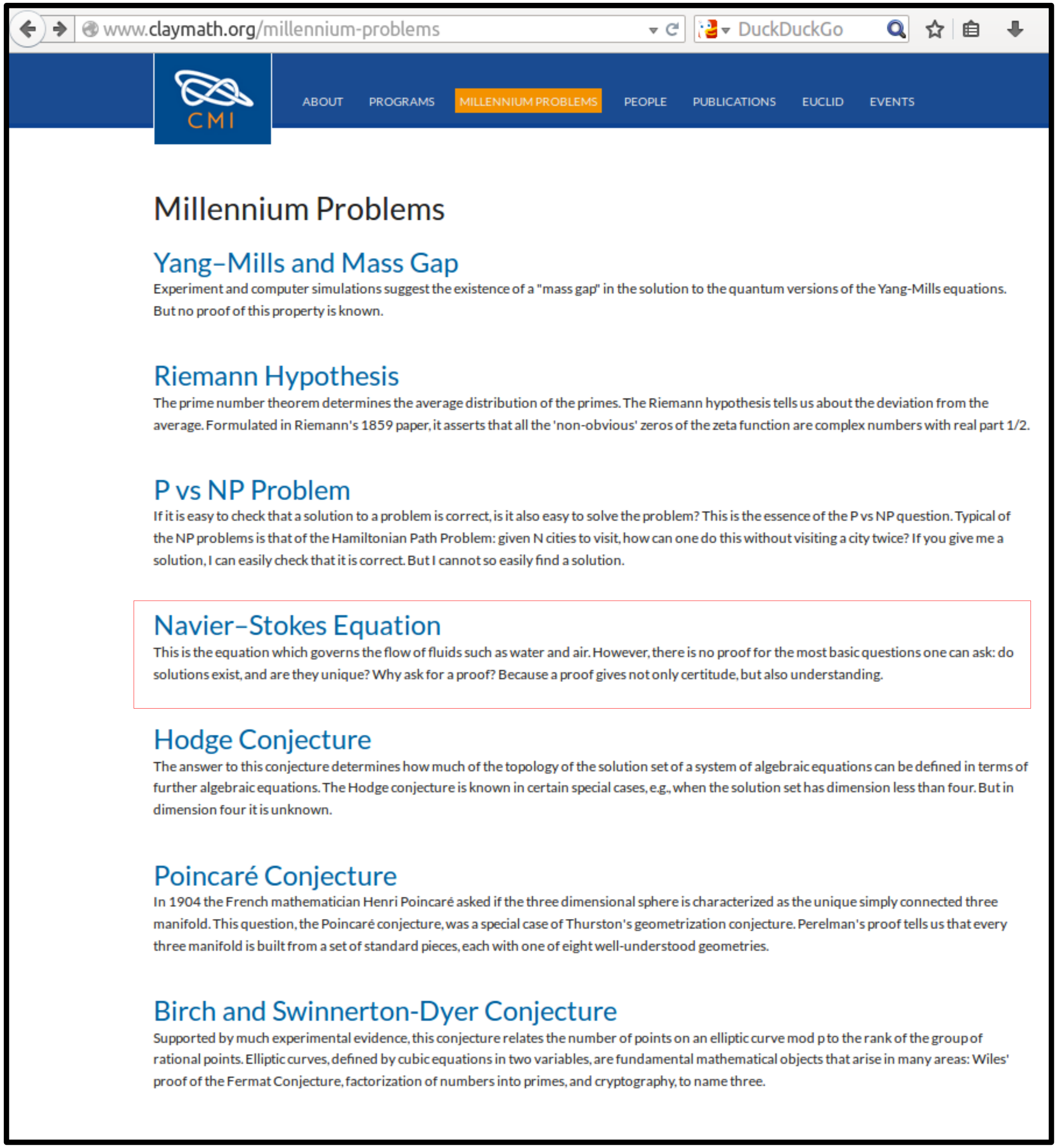
$$(\partial_t + \vec{v}\nabla)(P\rho^{-\gamma}) = 0$$

Adiabatic index γ = 5/3
for an ideal gas

Who wants to be a millionaire?

"For the three-dimensional system of equations, and given some initial conditions, mathematicians have not yet proved that smooth solution always exist"

-Wikipedia



Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

“During Alfvén's visit he gave a lecture at the University of Chicago, which was attended by Fermi. As Alfvén described his work, Fermi nodded his head and said, 'Of course.' The next day the entire world of physics said. 'Oh, of course. “ — Alex Dessler

Hannes Alfvén (1908 - 1995)

Maxwell Equations

$$\nabla \times \vec{E} = -\partial_t \vec{B}$$

Faraday's Law

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \underline{\frac{1}{c^2} \partial_t \vec{E}}$$

Ampere's Law

(displacement current can be neglected in non-relativistic plasmas)

$$\nabla \vec{E} = \underline{\frac{\rho_q}{\epsilon_0}}$$

Gauß Law

(Net charge density in plasmas usually zero)

$$\nabla \vec{B} = 0$$

Lorentz force

$$\vec{f} = \underline{\rho_q \vec{E}} + \vec{j} \times \vec{B}$$

Ohm's law

$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B})$$

Magnetohydrodynamics (MHD)

$$\partial_t \rho = -\nabla(\rho \vec{v})$$

Mass equation

$$(\partial_t + \vec{v}\nabla)\rho\vec{v} = -\nabla P + \vec{j} \times \vec{B}$$

Momentum equation

$$(\partial_t + \vec{v}\nabla)(P\rho^{-\gamma}) = 0$$

Energy equation

$$\partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{B}) + \frac{1}{\mu_0 \sigma}\nabla^2 \vec{B}$$

Induction equation

(Obtained by inserting Ohm's law and Ampere's law into Faraday's law)

These 8 equations determine ρ, **v**, P, **B**.

E and **j** are secondary variables derived from Ohm's and Ampere's law.

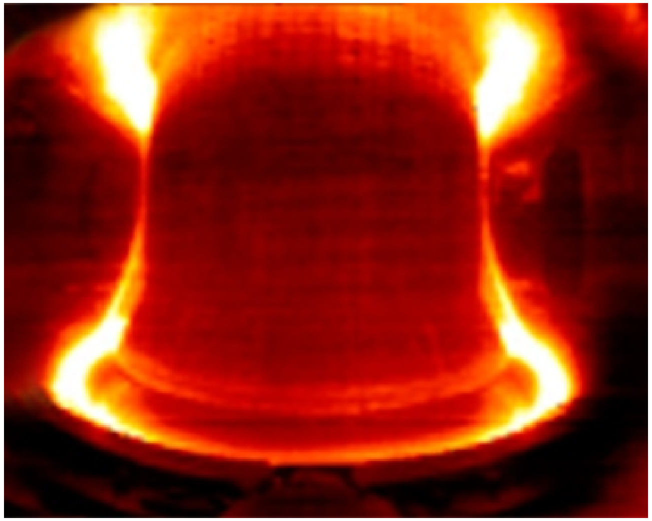
The Induction Equation

$$\partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \vec{B}$$

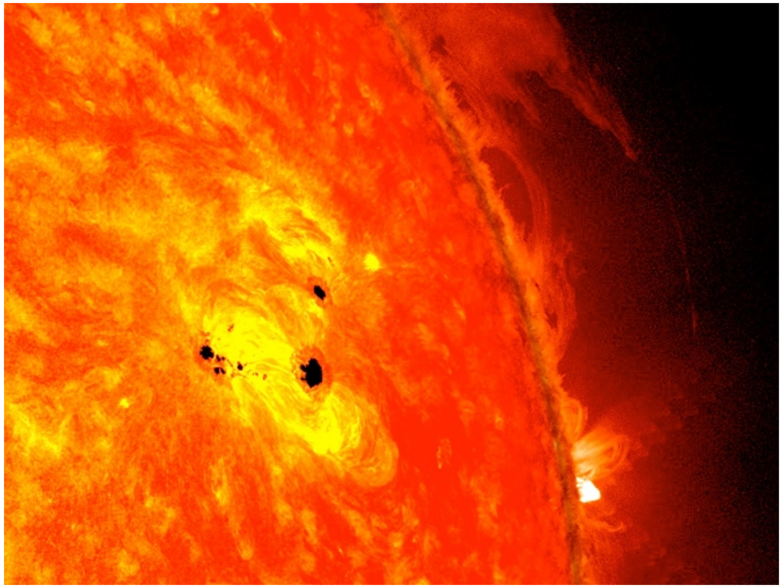
$$O(\frac{term1}{term2}) = R_m = \frac{vB/L}{B/(\mu_0 \sigma L^2)} = \frac{vL}{\eta}$$

<u>Magnetic Reynolds number</u>, with the <u>magnetic diffusivity</u> $\eta = \frac{1}{\mu_0 \sigma}$

For R_m >> 1 “<u>ideal MHD</u>” limit of perfect conductivity



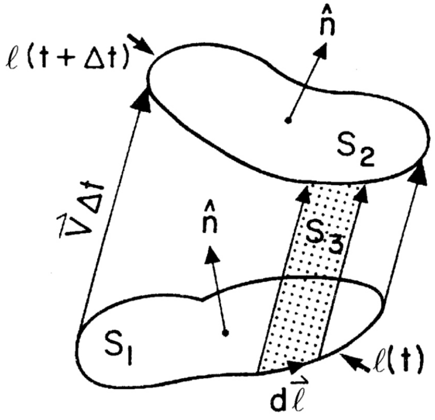
Substance	*L [m]*	*v [m/s]*	*η [m² /s]*	*R_m*
Laboratory Plasma	1	100	10	10
Earth's Core	1E+07	0.1	1	1E+06
Sun spot	1E+06	1E+04	1	1E+10
Interstellar Gas	1E+17	1E+03	1E+03	1E+17



Flux Freezing (or Alfvén's theorem)

How does the magnetic flux change on a moving plasma element?

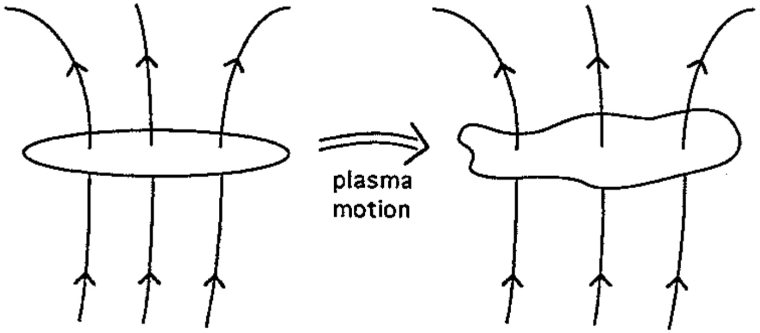
$$\phi_F = \int \vec{B} d\vec{A}$$

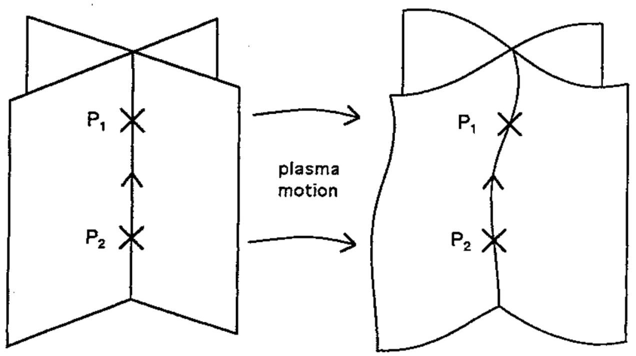


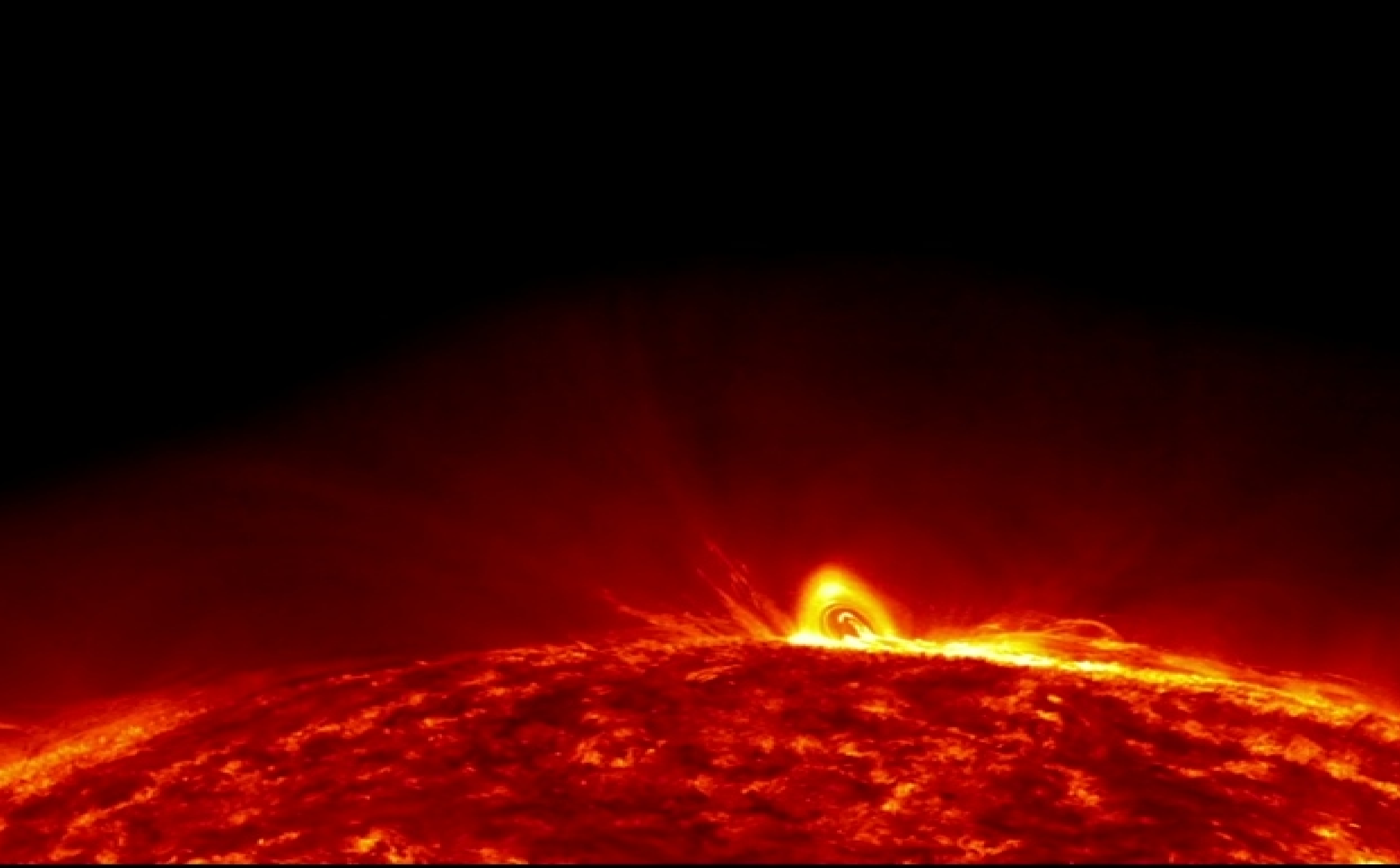
$$\frac{d\phi_F}{dt} = \int \partial_t \vec{B}\ d\vec{A} + \partial_t \oint \vec{B}(\vec{v}dt \times d\vec{s})$$

After applying Stokes theorem and vector identities:

$$\frac{d\phi_F}{dt} = \int (\partial_t \vec{B} - \nabla \times (\vec{v} \times \vec{B}))\ d\vec{A} = 0$$

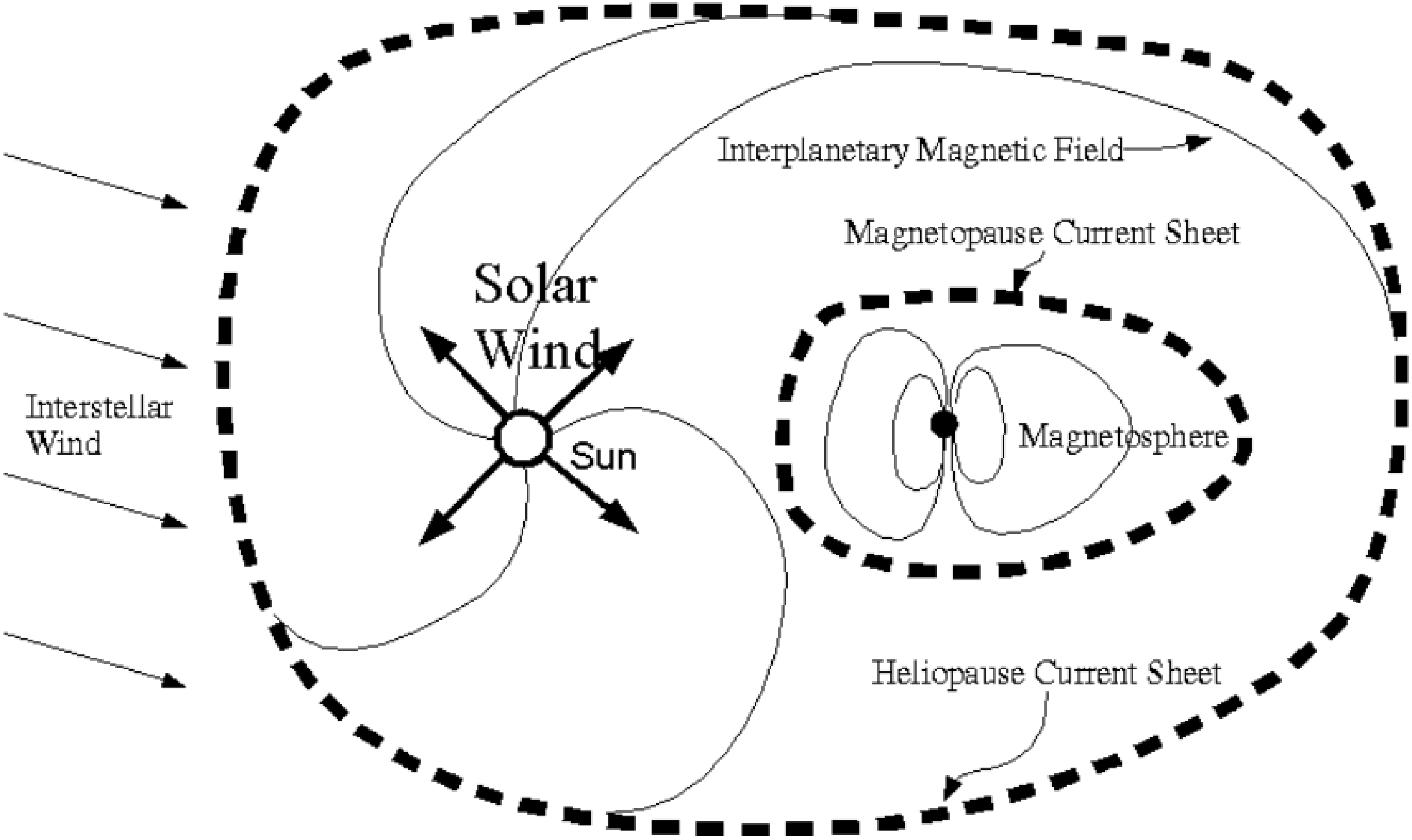




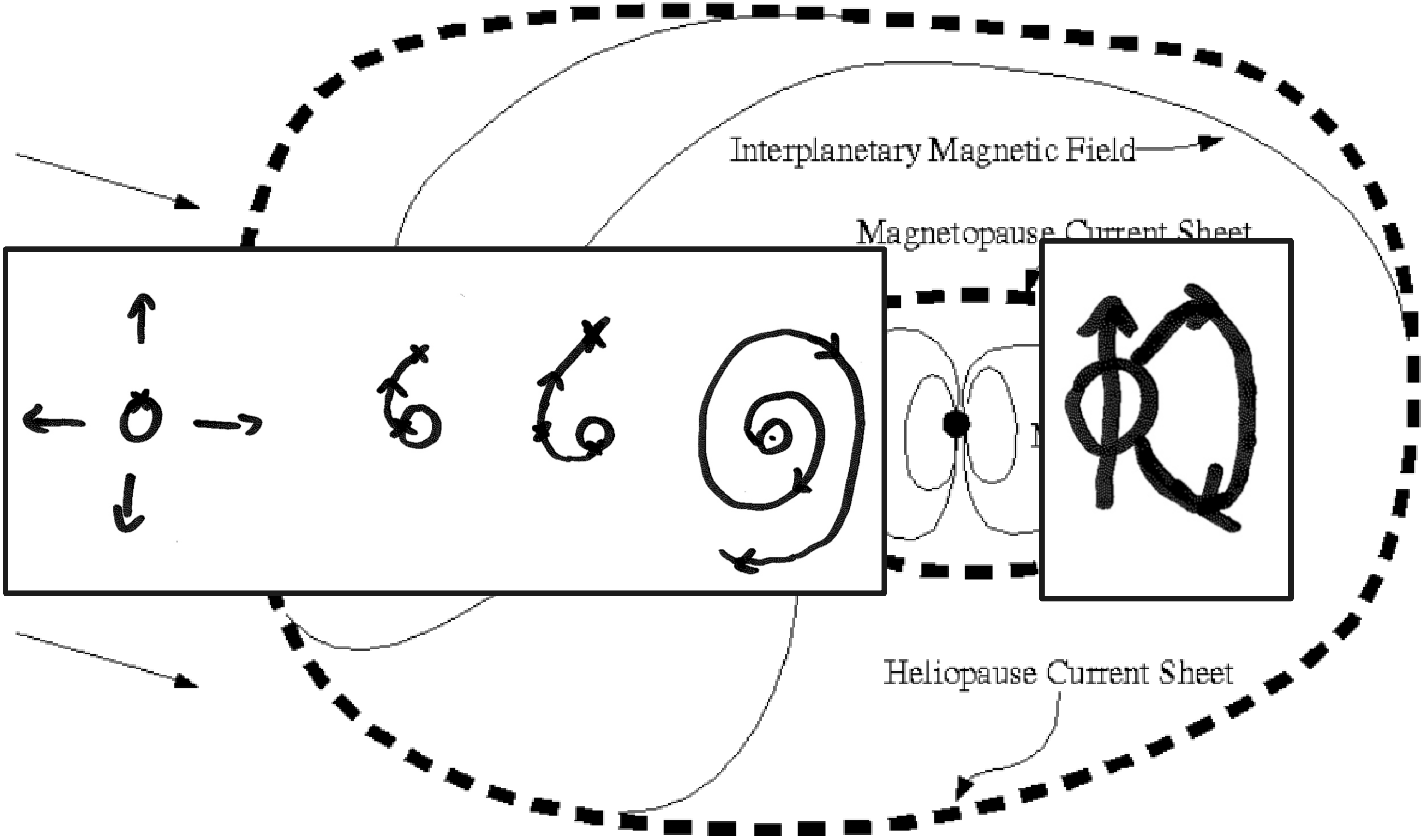


Solar flare in July 2012, credit NASA/SDO

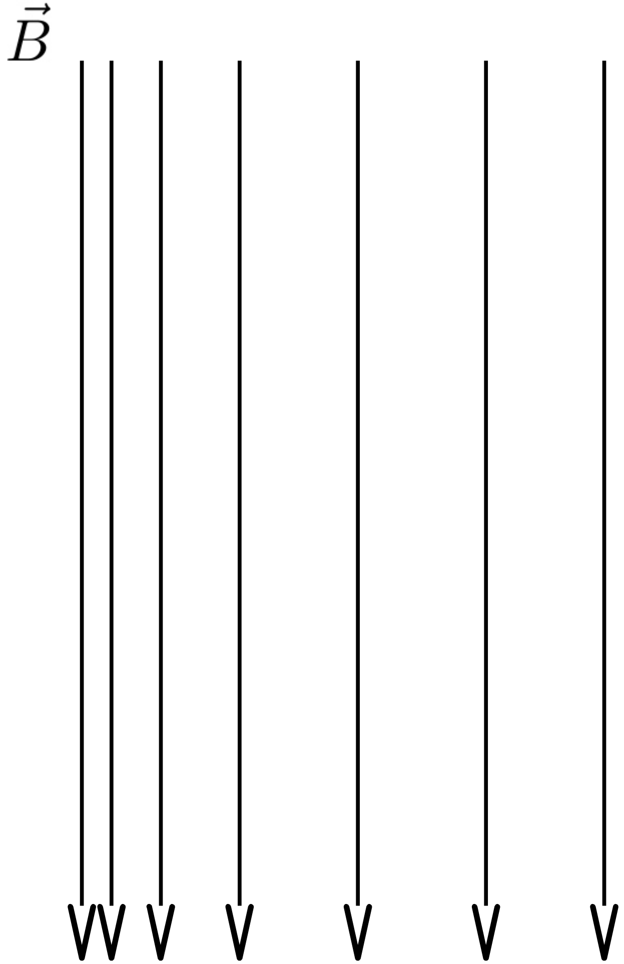
Universe of Plasma Bubbles



Solar Wind Magnetic Field



Effect of the Lorentz Force?



Inserting Ampere's law into
the Lorentz force:

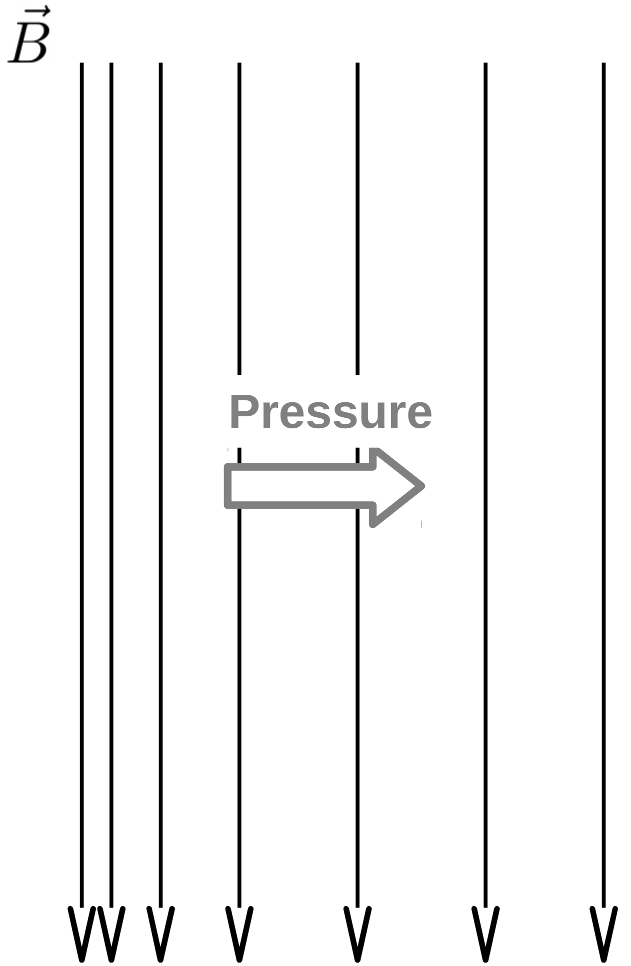
$$\vec{f} = \vec{j} \times \vec{B} = (\nabla \times \frac{\vec{B}}{\mu_0}) \times \vec{B}$$

$$\vec{f} = (\vec{B}\nabla)\frac{\vec{B}}{\mu_0} - \nabla(\frac{\vec{B}^2}{2\mu_0})$$

Tension

Pressure

Effect of the Lorentz Force?



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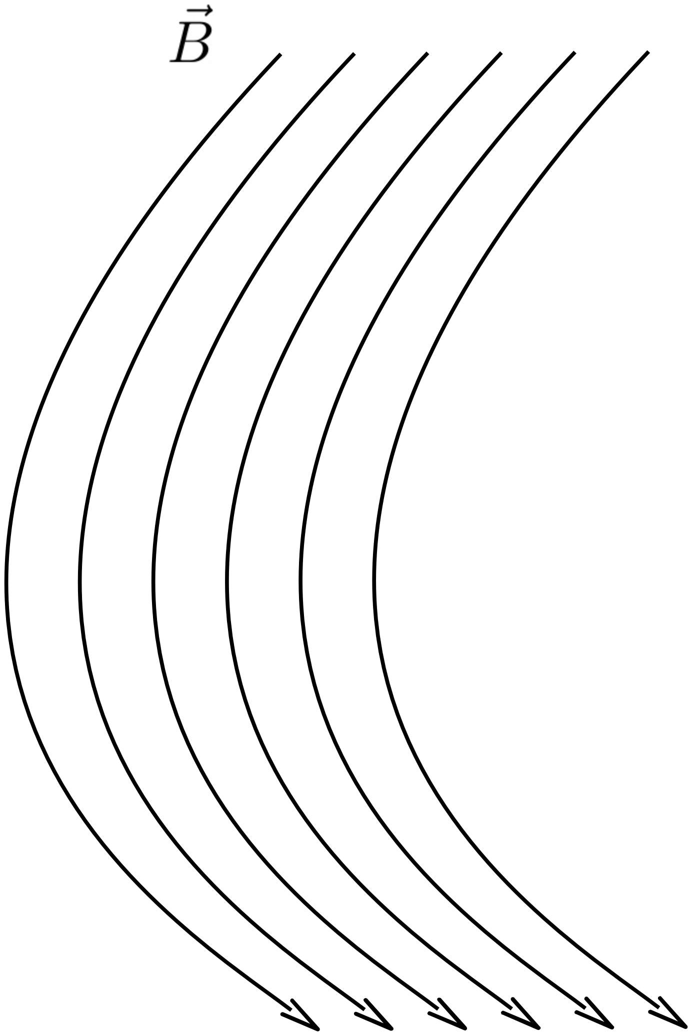

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<u>Tension</u>

<u>Pressure</u>

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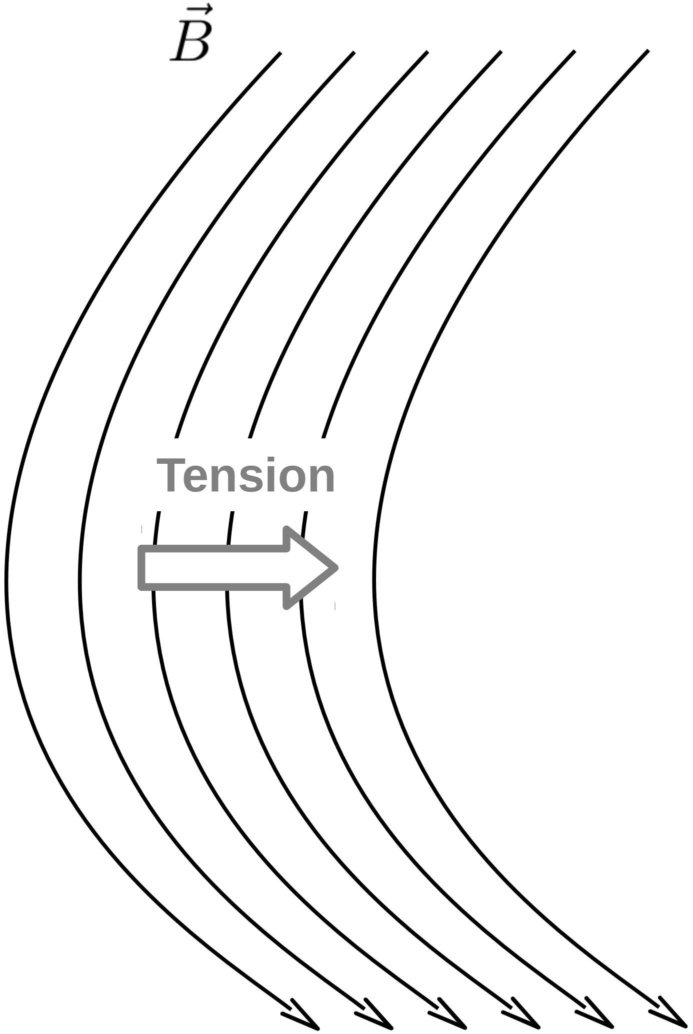

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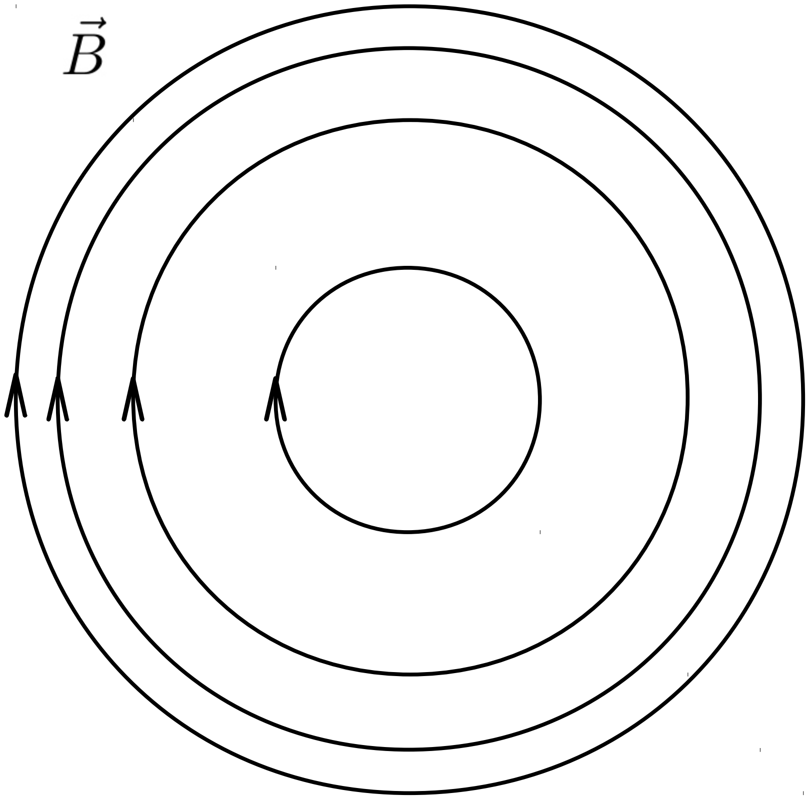

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<u>Tension</u>

<u>Pressure</u>

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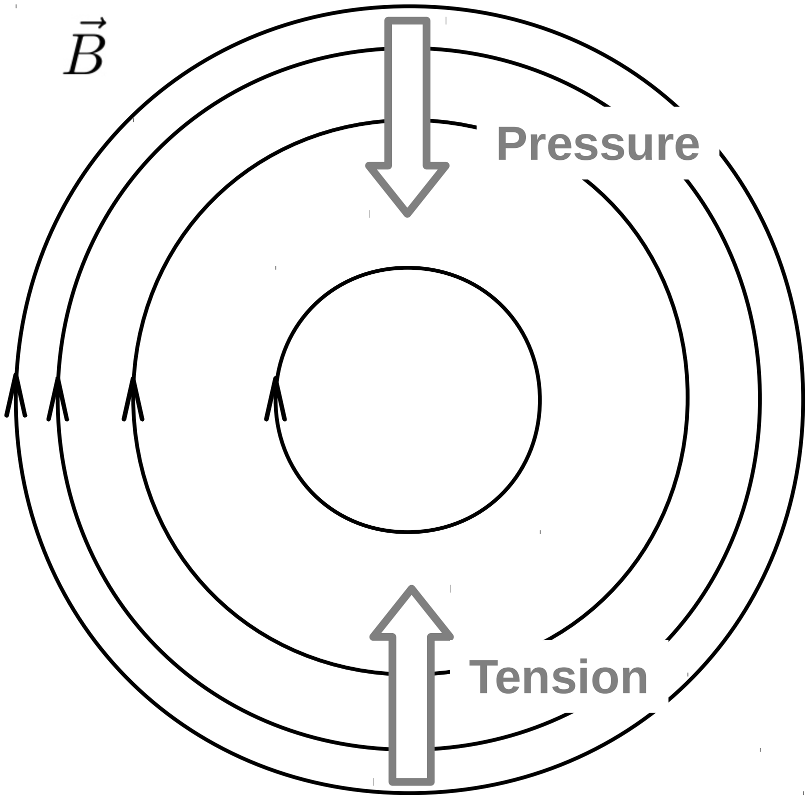

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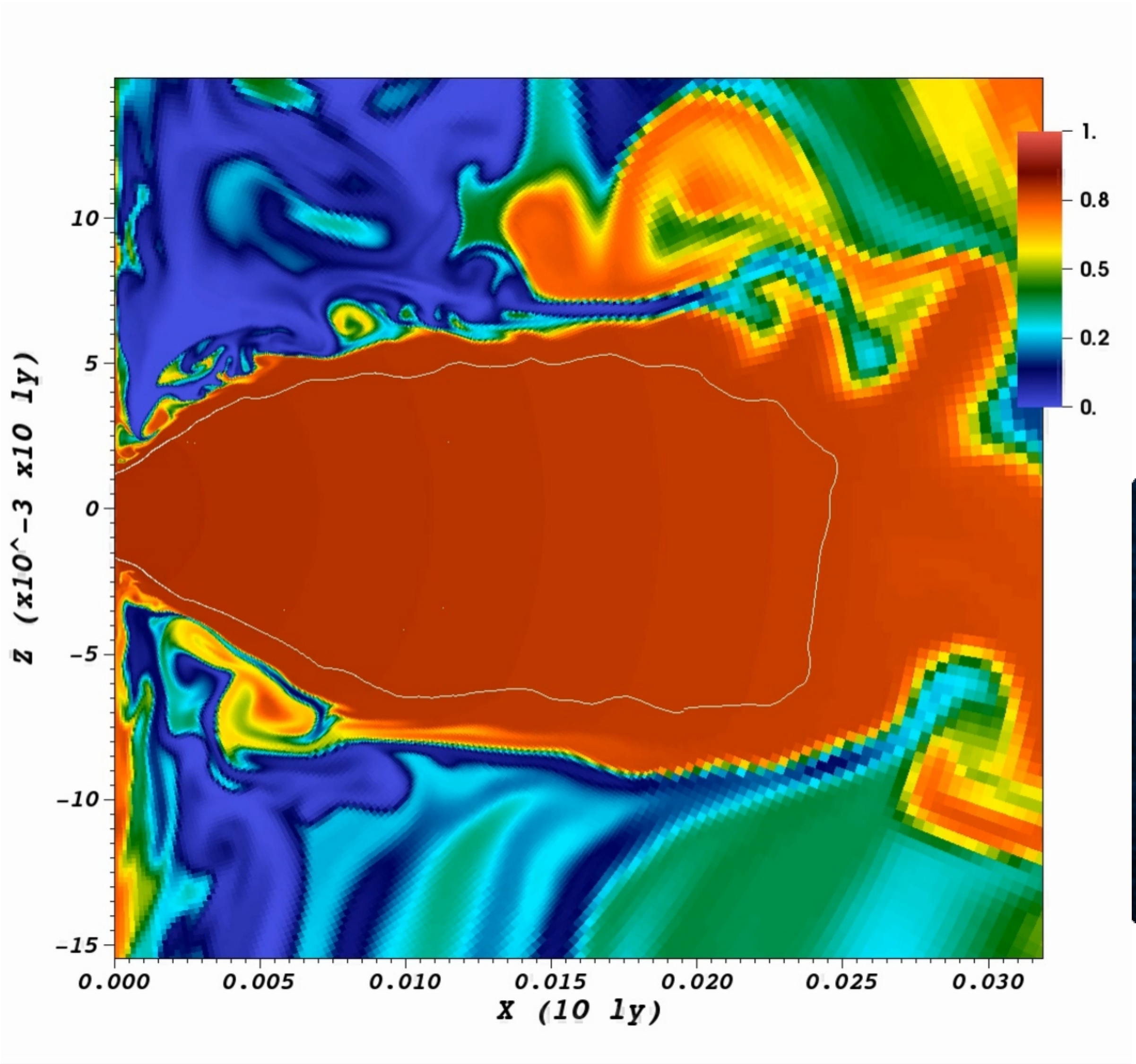
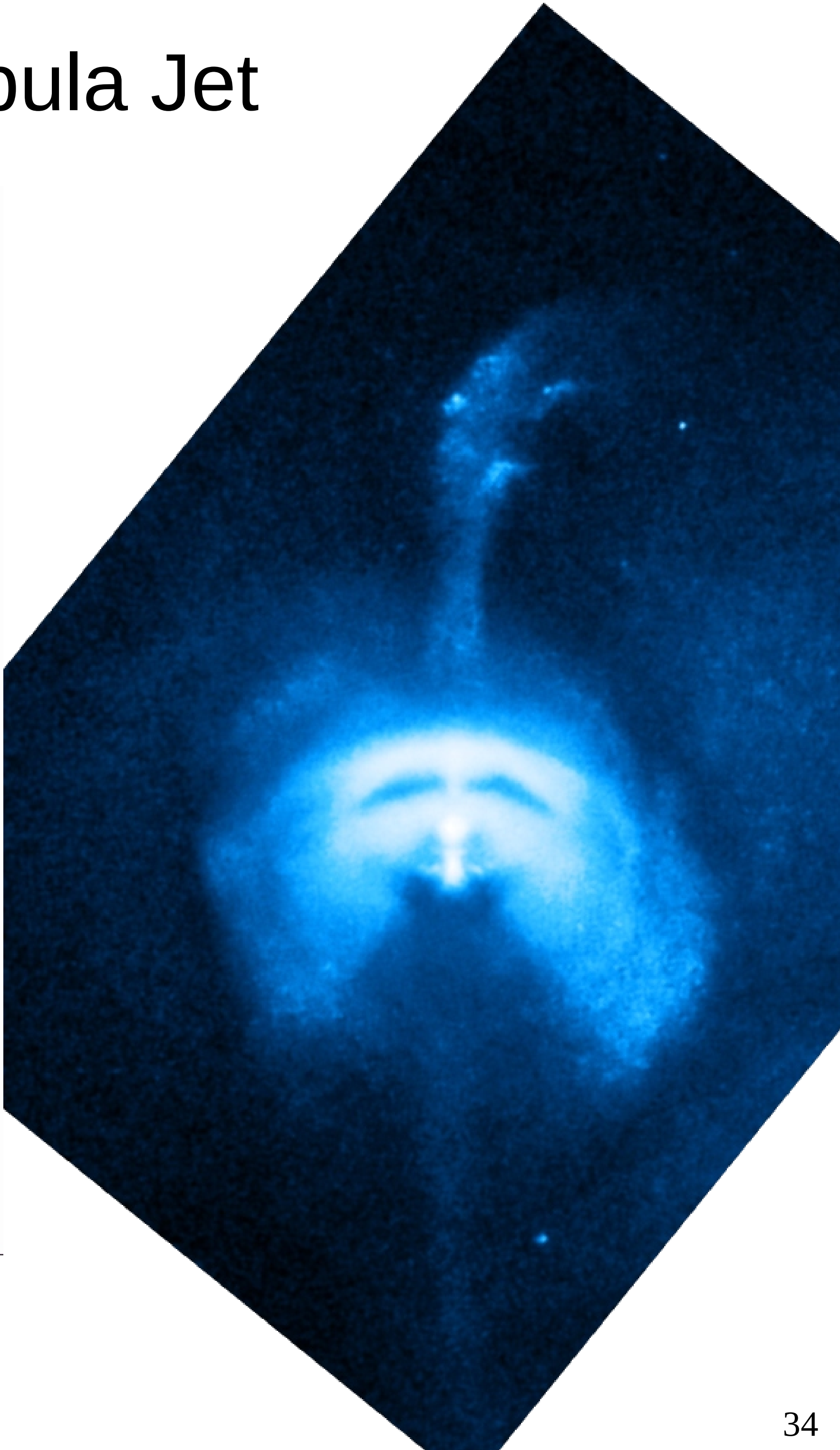
<u>Tension</u> <u>Pressure</u>

Pulsar Wind Nebula Jet

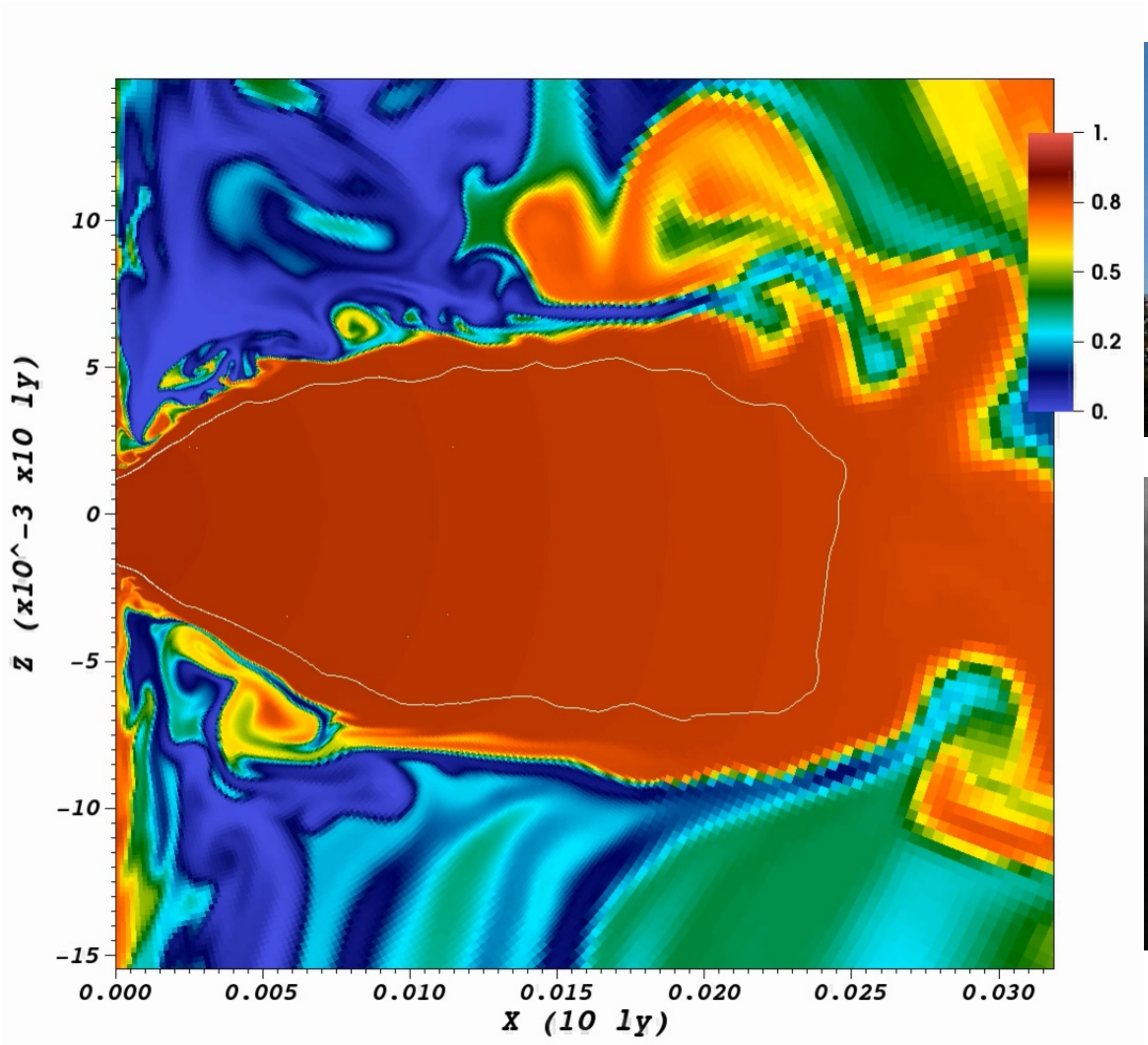


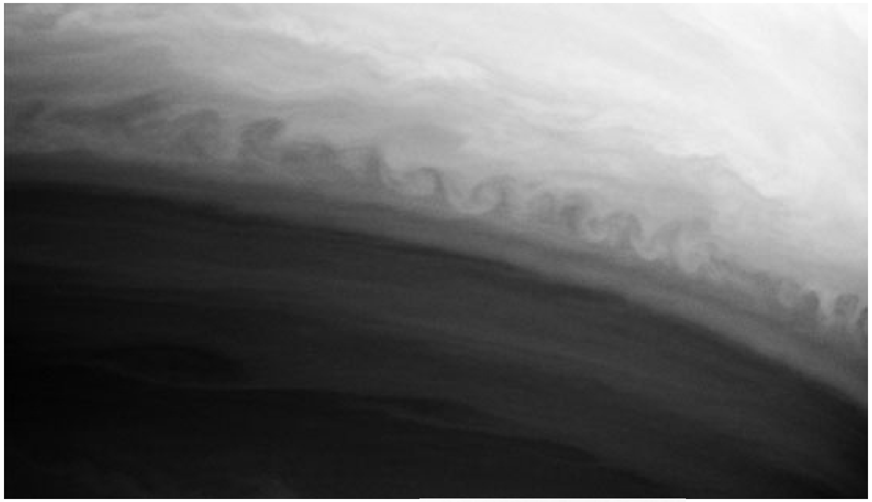


Fully toroidal magnetic field. Magnetic tension pushes plasma back on the axis.

Buehler and Giomi, MNRAS 462 3, 2016

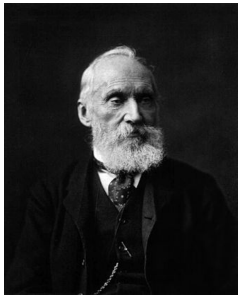
Pulsar Wind Nebula Jet





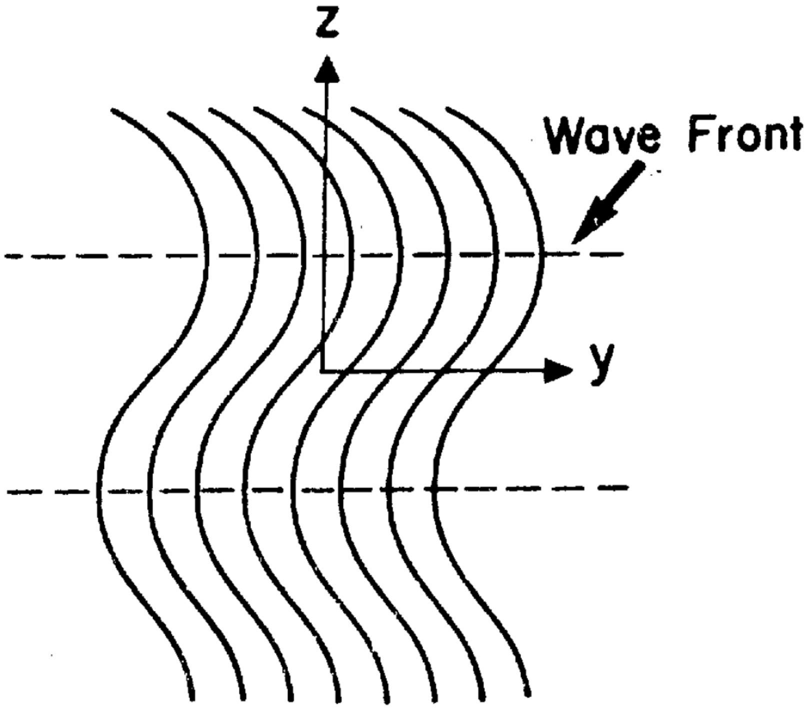
Kelvin-Helmholtz instability
at shear flow

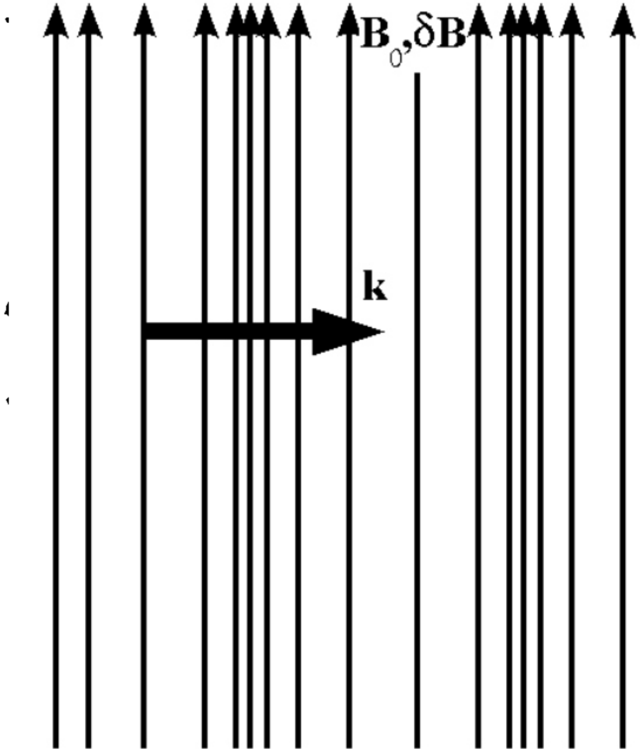
Buehler and Giomi, MNRAS 462 3, 2016

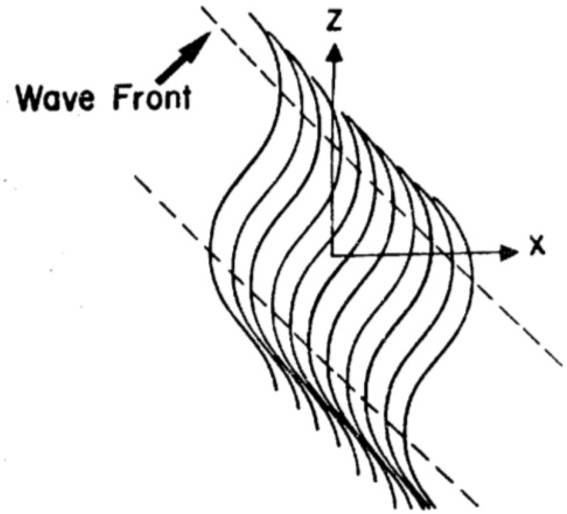


MHD Waves

Three kinds of wave solutions: Alfvén waves (due to tension), fast waves and slow waves (both due to compression). Their velocities depend on direction.

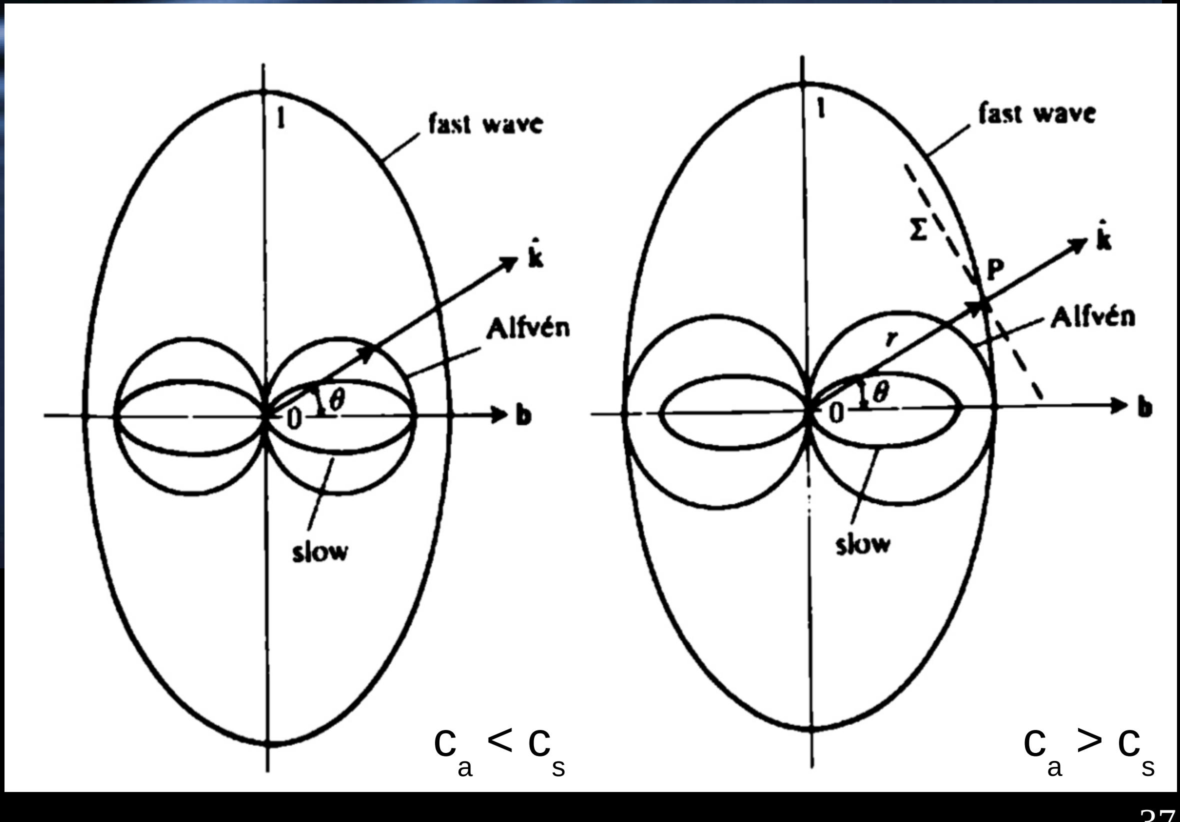






Sound speed: $$c_s = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

Alfvén speed: $$c_a = \sqrt{\frac{B_0^2}{\mu_0 \rho_0}}$$


1
fast wave
$\hat{k}$
Alfvén
θ
0
b
slow
1
fast wave
Σ
P
$\hat{k}$
Alfvén
r
θ
0
b
slow

$c_a < c_s$

$c_a > c_s$

Summary

- The Universe can be thought of as bubbles of plasma.
- MHD combines Hydrodynamics with Electrodynamics, approximately describes plasmas on large scales (large magnetic Reynolds number)
- Flux freezing follows for ideal MHD (close to zero resistivity). Allows to understand plasma behavior intuitively on large scales.
- Linearization of the equations leads to 3 MHD waves. Their speed is direction dependent and is characteristically the Alfvén speed.

For good introductions see for example:
- S. J. Schwartz. Astrophysical Plasmas http://www.sp.ph.imperial.ac.uk/~sjs/
- H. Spruit. Essential Magnetohydrodynamics for Astrophysics arXiv:1301.5572

Thank you for your attention..