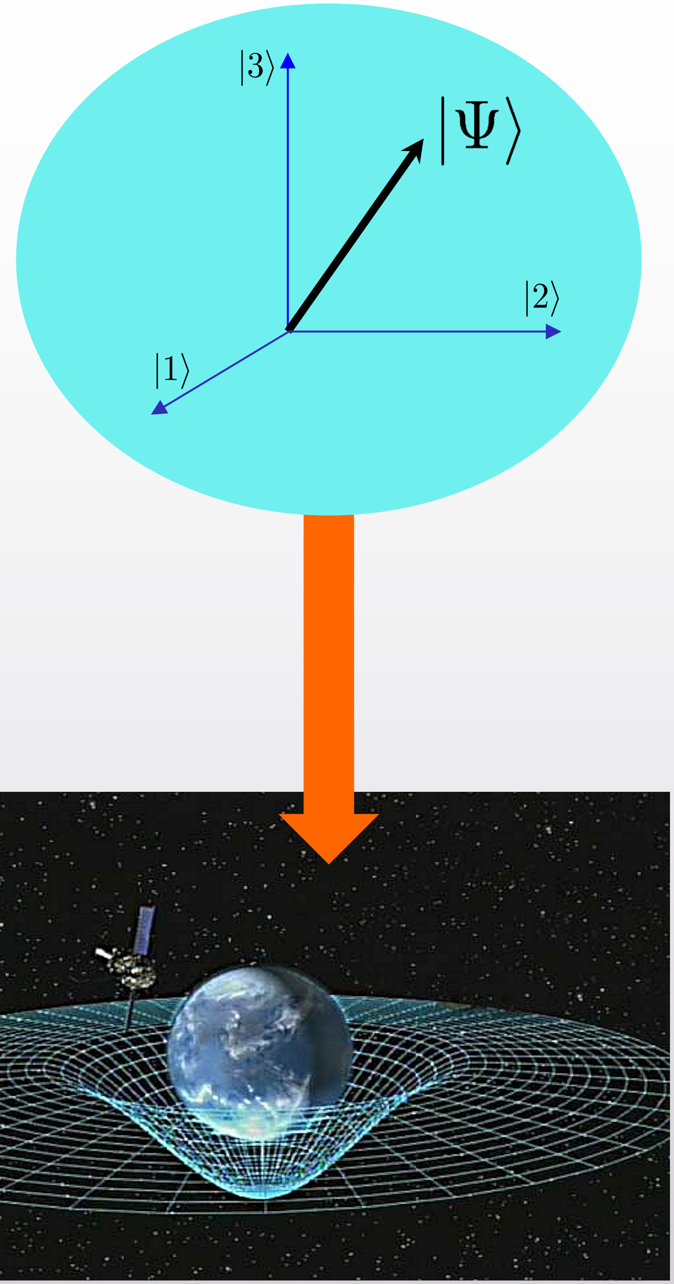
From Quantum Mechanics to Spacetime



Sean Carroll
Caltech & Santa Fe Institute

How physicists generally teach and think:

Classical model → *Quantum theory*

What we're going to try:

Quantum theory → **Classical world**

Gravity in particular is not a standard quantum field theory.

- Finite number of degrees of freedom
- Nonlocal effects (holography, black hole info)

Don't quantize gravity;
find gravity w/in quantum mechanics.

How physicists typically invent quantum theories

Start classically, e.g. nonrelativistic particle in 1 dimension:

Coordinate x, momentum p (together: phase space),

Hamiltonian $H(x,p) = \frac{p^2}{2m} + V(x)$.

Promote to a wave function, a square-integrable complex function of the coordinate (or the momentum, not both), obeying Schrödinger's equation:

$$\psi : \mathbb{R} \to \mathbb{C}, \qquad \int \psi^* \psi \, dx < \infty.$$

$$\hat{H}\psi = i\hbar \frac{\partial}{\partial t}\psi, \qquad \hat{p} = -i\hbar \frac{\partial}{\partial x}$$

These wave functions form a Hilbert space, $\mathcal{H}$:
a complete, complex, normed vector space.

$$\psi(x) \to |\psi\rangle \in \mathcal{H}, \qquad \alpha|\psi\rangle + \beta|\phi\rangle \in \mathcal{H}, \qquad \langle\psi|\phi\rangle \in \mathbb{C}$$

A quantum theory is therefore defined by:

1) a choice of Hilbert space, which is just its dimension:

$$d = \dim \mathcal{H}$$

2) a Hamiltonian $\hat{H}$, powering the Schrödinger equation:

$$\hat{H}|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle$$

Technicality: what about observables?

They are represented by self-adjoint operators,

$$\hat{\mathcal{O}}_i : \mathcal{H} \to \mathcal{H}, \quad \langle \hat{\mathcal{O}}_i \psi | \phi \rangle = \langle \psi | \hat{\mathcal{O}}_i \phi \rangle.$$

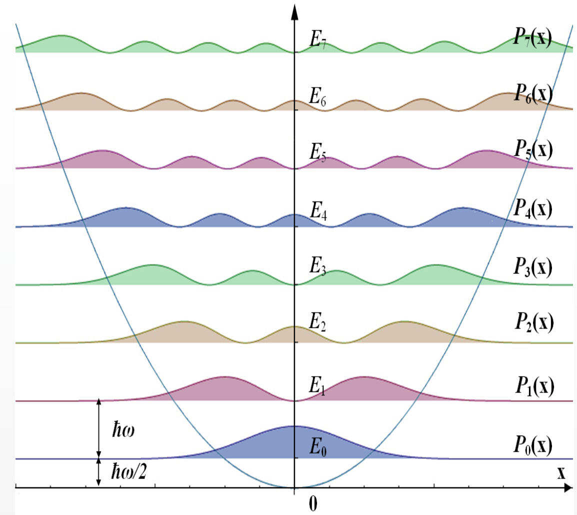
and specified by an "algebra":

$$[\hat{\mathcal{O}}_i, \hat{\mathcal{O}}_j] = f_{ij}^k \hat{\mathcal{O}}_k.$$

If $\dim \mathcal{H}$ is finite, all Hermitian operators ($\hat{\mathcal{O}}^\dagger = \hat{\mathcal{O}}$) are self-adjoint, and define good observables.

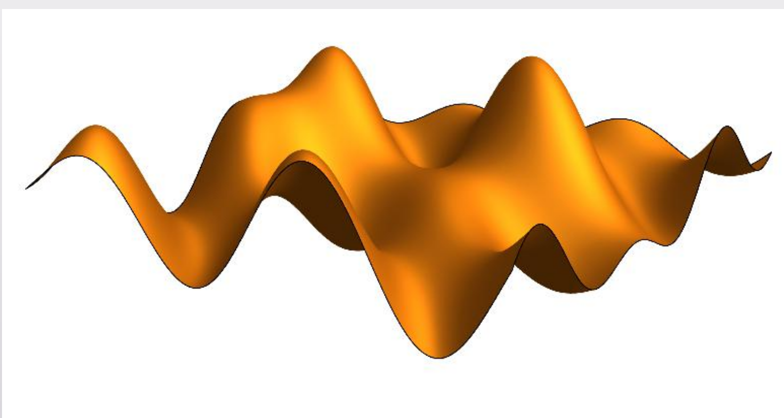
But if $\dim \mathcal{H} = \infty$, different sets of observables define different quantum theories.

In standard non-relativistic QM, Hilbert space is infinite-dimensional.



E.g. harmonic oscillator: there are an infinite number of energy levels.

Likewise in Quantum Field Theory, $\dim \mathcal{H} = \infty$.



Each fixed-wavelength mode acts like a harmonic oscillator.

Even with short- and long-distance cutoffs, each mode has $\dim \mathcal{H} = \infty$.

Gravity changes everything

Consider a region of size L.
If you put too much energy into it (e.g. by exciting modes of a quantum field), it all collapses into a black hole.

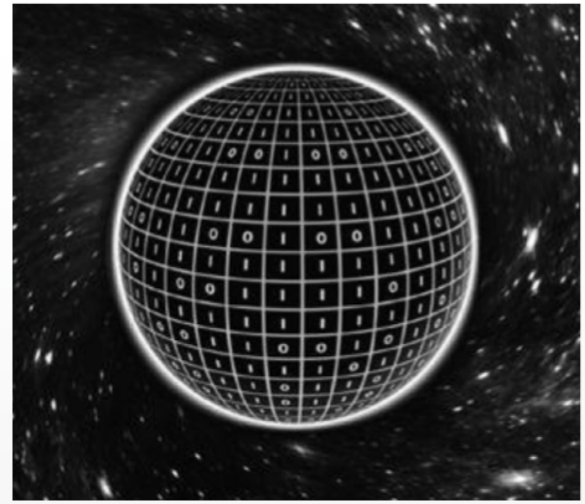
If you put even more energy in, the black hole grows to be larger than L.

A finite number of states suffice to describe anything that can happen in a region. Locally, $\dim \mathcal{H} = $ finite.

Corollary: gravity is not a quantum field theory.

[Bekenstein; Bousso; Bao, Carroll & Singh, 2017]

We even have an idea what $\dim \mathcal{H}$ is. Black holes have entropy proportional to the area of the event horizon:



$$S_{BH} = \frac{A}{4G} \sim \left(\frac{L}{L_P}\right)^2$$

For systems at maximum entropy, the dimensionality of Hilbert space is the exponential of that entropy:

$$\dim(\mathcal{H}) \sim e^S \sim e^{(L/L_P)^2}.$$

Big, but finite! For one cubic centimeter of space,

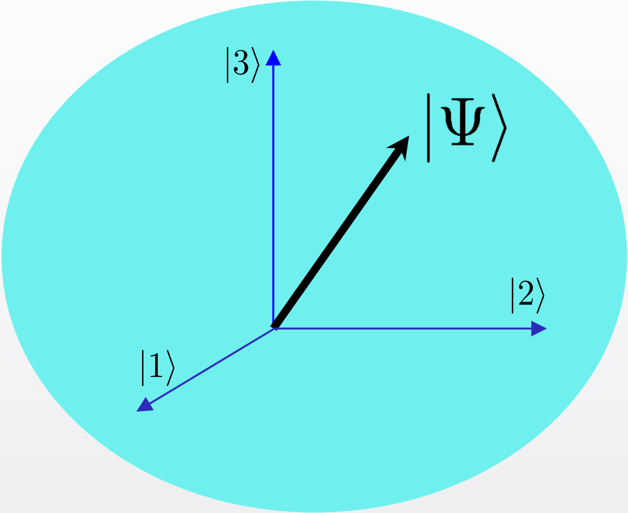
$$\dim(\mathcal{H}_{\mathrm{cm}^3}) \sim e^{10^{66}}.$$

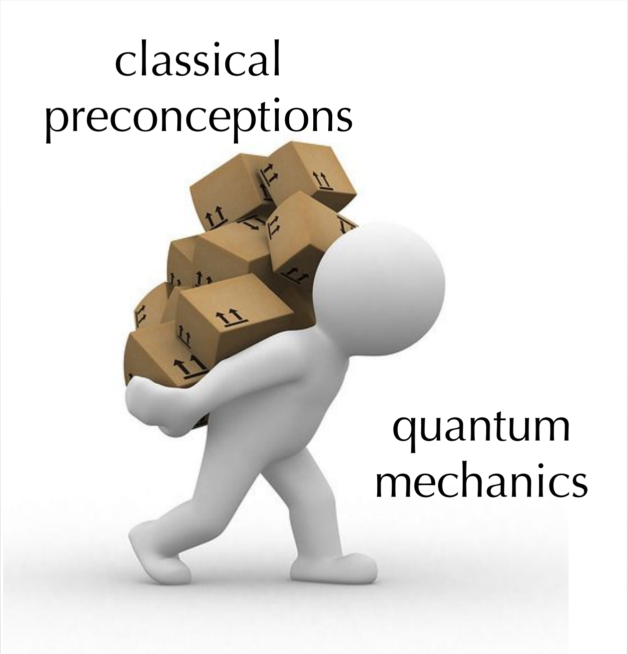
Observables are just "all Hermitian operators."

So in our quest *Quantum theory* → **Classical world** ,
we have very little to work with!

The universe is described by
a vector in Hilbert space obeying
the Schrödinger equation

$$\hat{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$$





The elements of standard classical reality
(space, fields, particles, forces) must
emerge from Hilbert-space concepts:
vectors, Hamiltonian, density matrices,
tensor products, entropy, entanglement…

[cf. Zurek; Hartle; Giddings]

You might think we have the *form* of the Hamiltonian. E.g. for a harmonic oscillator:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2\hat{x}^2$$

But that's specifically in the position basis, $\hat{x}|x\rangle = x|x\rangle$. How did we know to use that?

All we really have is the *spectrum* of the Hamiltonian: the set of energy eigenvalues.

$$\hat{H}|n\rangle = E_n|n\rangle$$

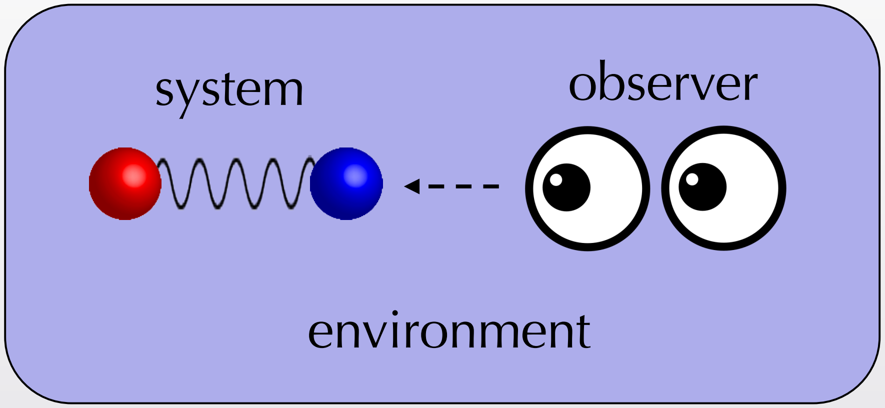
Our task: reconstruct the whole world from a set of real numbers $\{E_n\}$, the eigenvalues of $\hat{H}$.

[Carroll & Singh, 2018]

How do we go from a list of energy eigenvalues to an interpretation in terms of physical structures?

Well, where did the position basis come from, anyway?

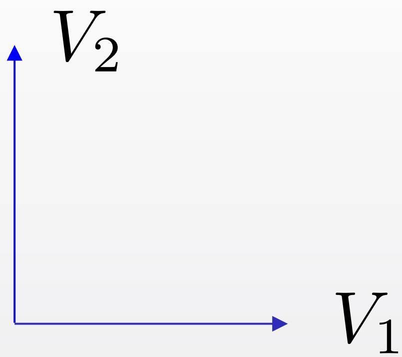
Intuitively, positions are what we see when we look at things.



More formally: there are many ways to decompose Hilbert space into interacting subsystems. *Useful* decompositions are those that correspond to notions familiar from our experience with the classical world.

Quantum Mereology:
Carving Hilbert Space into Subsystems

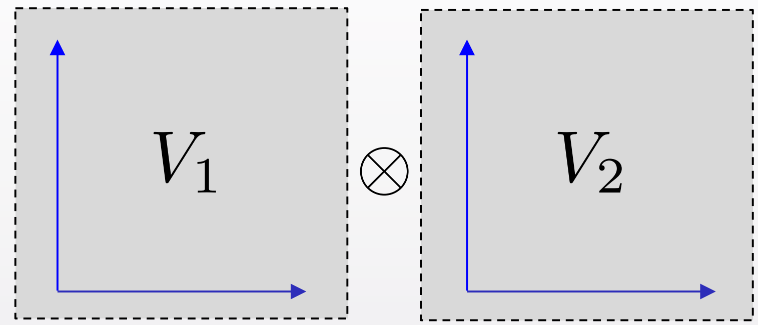
Direct sums are "or".



$$\vec{v} \oplus \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ w_1 \\ w_2 \\ \vdots \end{pmatrix}$$

$$\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2$$

Tensor products are "and".



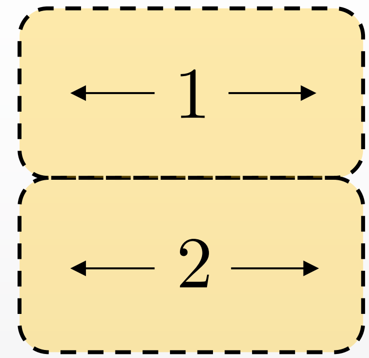
$$\vec{v} \otimes \vec{w} = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ \vdots \\ v_2 w_1 \\ v_2 w_2 \\ \vdots \end{pmatrix}$$

$$\dim(V_1 \otimes V_2) = \dim V_1 \times \dim V_2$$

Tensor factorizations are choices of basis

Consider two qubits,

$$\mathcal{H} = \{|\uparrow\rangle_1, |\downarrow\rangle_1\} \otimes \{|\uparrow\rangle_2, |\downarrow\rangle_2\}.$$



Alternative choice of basis:

$$\mathcal{H} = \{|0\rangle_\alpha, |1\rangle_\alpha\} \otimes \{|0\rangle_\beta, |1\rangle_\beta\}$$

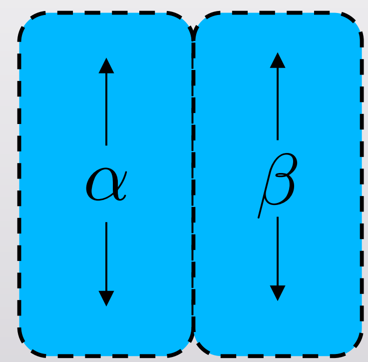
such that

$$|0\rangle_\alpha |0\rangle_\beta = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\uparrow\rangle_2 + |\downarrow\rangle_1 |\downarrow\rangle_2 \right)$$

$$|0\rangle_\alpha |1\rangle_\beta = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\uparrow\rangle_2 - |\downarrow\rangle_1 |\downarrow\rangle_2 \right)$$

$$|1\rangle_\alpha |0\rangle_\beta = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2 \right)$$

$$|1\rangle_\alpha |1\rangle_\beta = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \right)$$



How does the Hamiltonian help us choose subsystems?

$$\hat{H} : \mathcal{H} \to \mathcal{H}$$

Decomposing Hilbert space into a tensor product implies a decomposition of the Hamiltonian.

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \;\; \to$$

$$\hat{H} = \hat{H}_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \hat{H}_B + \hat{H}_{\text{int}}$$

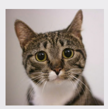
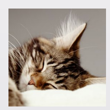
self-Hamiltonian for subsystem A	self-Hamiltonian for subsystem B	interaction Hamiltonian

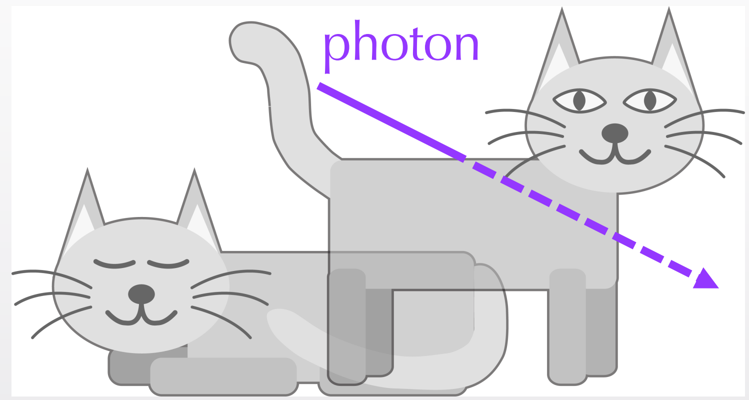
What is the **most useful way** to factor Hilbert space into a set of interacting subsystems?

Step One: Classical Behavior

Classical systems have localized wave functions that approximately obey classical equations (Ehrenfest).

They also remain unentangled with the environment. Classical systems take preferred pointer states, not Schrödinger-cat states

$\frac{1}{\sqrt{2}}\big(|$  $\rangle + |$  $\rangle\big)$

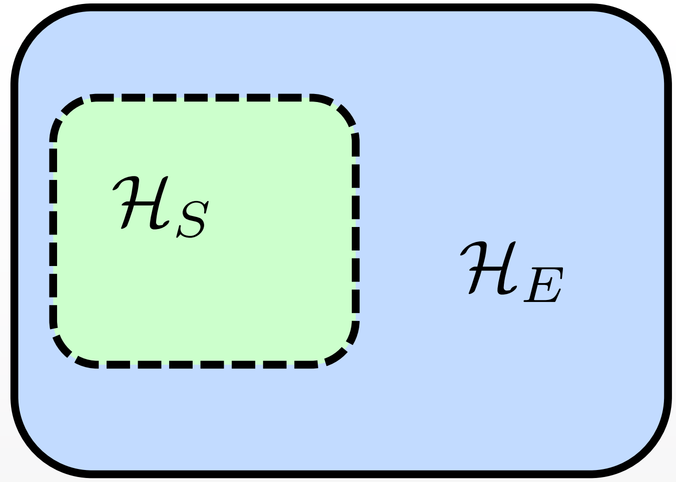


decoherence

So two criteria for classicality:

1. Localized states remain localized, and
2. Unentangled states remain unentangled.

Given some factorization $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ we can write the Hamiltonian as



$$\hat{H} = \hat{H}_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes \hat{H}_E + \hat{H}_I.$$

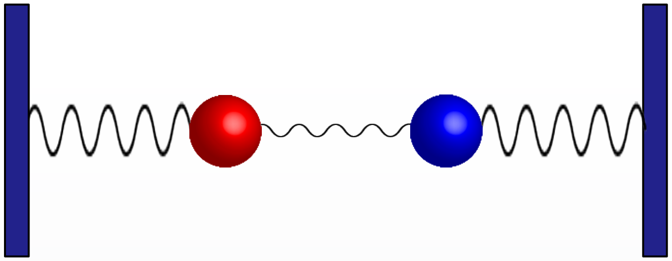
System Hamiltonian governs "remains localized"

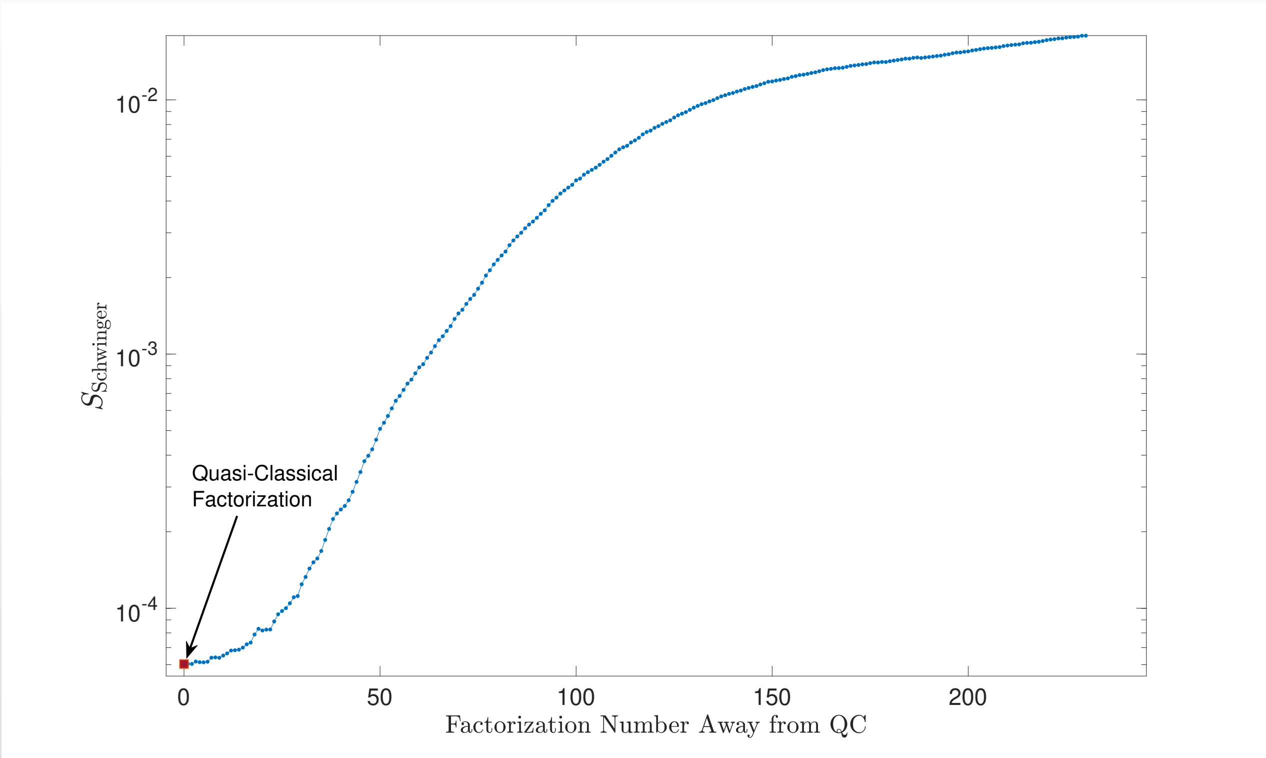
Interaction Hamiltonian governs "remains unentangled"

The algorithm is: given $\hat{H}$, sift through all possible factorizations $\mathcal{H}_S \otimes \mathcal{H}_E$ that simultaneously minimize spread of the system wave function and entanglement with the environment.

[Carroll & Singh, 2020]

Example where we start with a known system (coupled oscillators), change to a different factorization, and verify that our criterion pinpoints the original one.

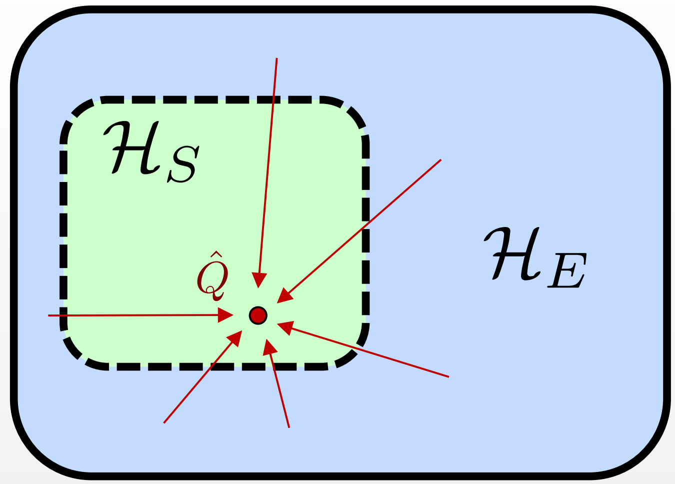




Upshot: we can split Hilbert space into system/environment (or other subsystems) purely from the Hamiltonian.

Why don't we live in momentum space?

Entanglement grows slowly when the environment monitors a particular "pointer observable" in the system. Call it $\hat{Q}$. It then defines a conjugate observable. Call it $\hat{P}$.



Then the system remains localized if the system Hamiltonian takes the form

$$\hat{H}_S = \hat{P}^2 + V(\hat{Q}).$$

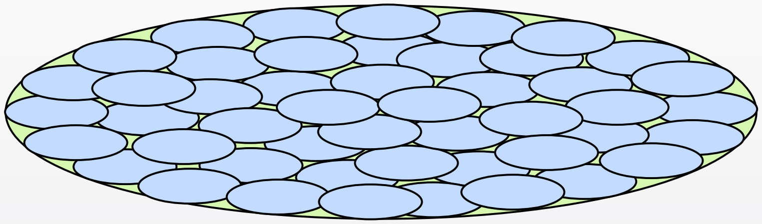
Maybe "position" is special in classical mechanics because classical limits of quantum theories have this form?

[Carroll & Singh, 202?]

Step Two: Space and Locality

Consider a Hilbert space factored into finite-dim pieces.
Can we factor to represent local regions of space?

$$\mathcal{H} = \bigotimes_a \mathcal{H}_a$$



We can write the Hamiltonian as a sum of operators connecting all possible combinations of factors.

$$\hat{H} = \sum_a h_a \hat{\mathcal{O}}_a^{(\mathrm{self})} + \sum_{ab} h_{ab} \hat{\mathcal{O}}_{ab}^{(2-\mathrm{pt})} + \sum_{abc} h_{abc} \hat{\mathcal{O}}_{abc}^{(3-\mathrm{pt})} + \cdots$$

A "k-local Hamiltonian" is one where any factor only interacts with k other factors, its "nearest neighbors."

Cotler, Penington, and Renard, 2017:

- Most Hamiltonians have no local factorization.
- When it exists, locality is (mostly) unique.

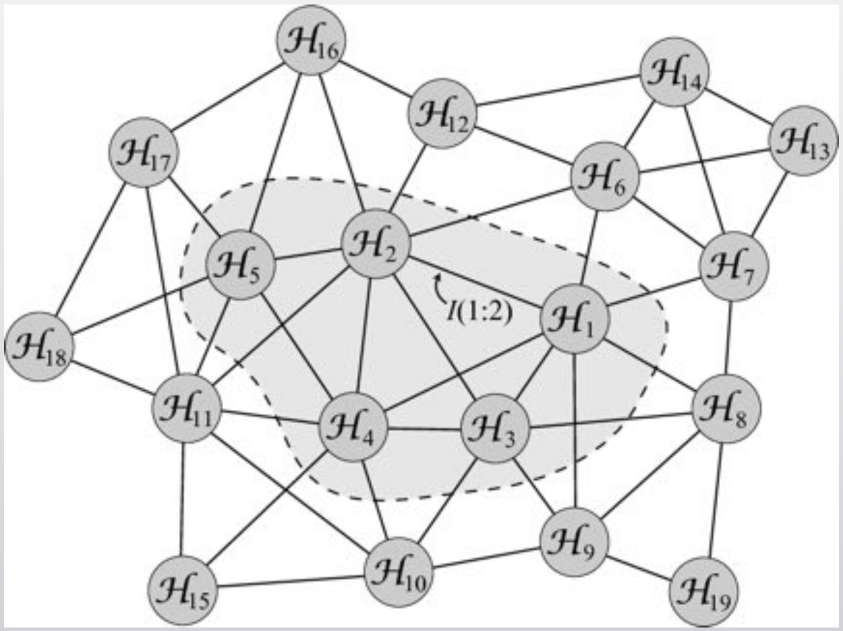
n^k $<<$ 2^n

parameters in a local Hamiltonian

energy eigenvalues (dim $\mathcal{H}$)

This gives a topology to the graph made from Hilbert subspaces, with edges defined by interactions.

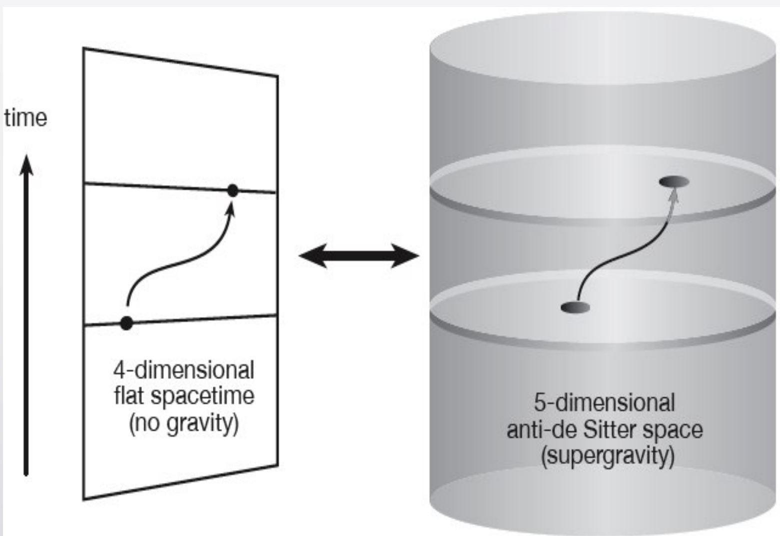
$$\mathcal{H} = \bigotimes_a \mathcal{H}_a$$



Step Three: Gravity and Curved Spacetime

"Local interactions" are a fixed structure that defines the dimensionality & topology of space.

Geometry is dynamical, it depends on the state.
What could define that? Entanglement.



AdS/CFT: entanglement in boundary field theory defines geometry in the bulk.

(Note: not a quantum theory "of" one specific thing.)

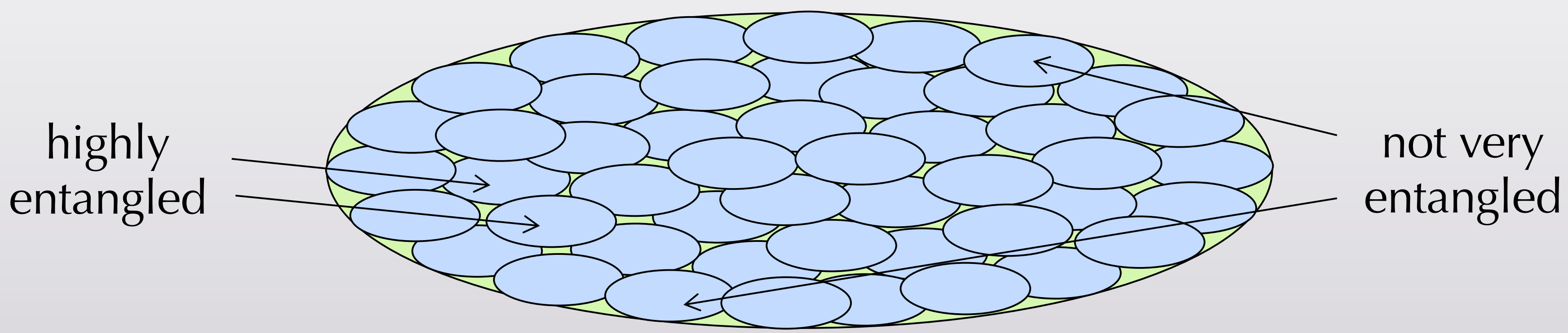
Still worth asking why apples fall from trees.

[cf. Swingle 2009; Van Raamsdonk 2010; Evenbly & Vidal 2011; Maldacena & Susskind 2014]

Can we work directly in our "bulk"?

Forget about black holes/AdS, think about nearly-empty space, where gravity appears local.

In quantum field theory, empty space is a busy place. Modes in a nearby regions are highly entangled, and the closer they are to each other, the more entangled they get.

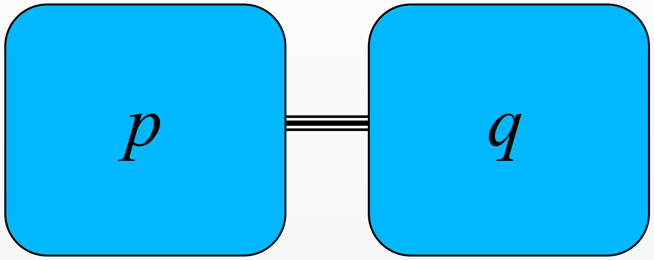


Turn this around! *Define* "nearby" as "highly entangled."

[Cao, Carroll & Mikhalakis 2016]

Quantifying entanglement

von Neumann: entanglement between two subsystems is implies entropy.

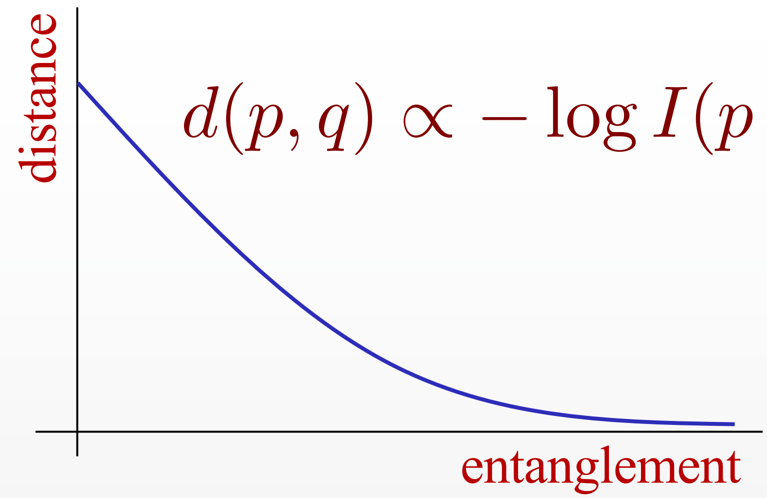


$$\hat{\rho}_p = \mathrm{Tr}_q \, |\Psi\rangle \langle\Psi| \, , \quad S_p = -\mathrm{Tr}_p \, \hat{\rho}_p \log \hat{\rho}_p$$

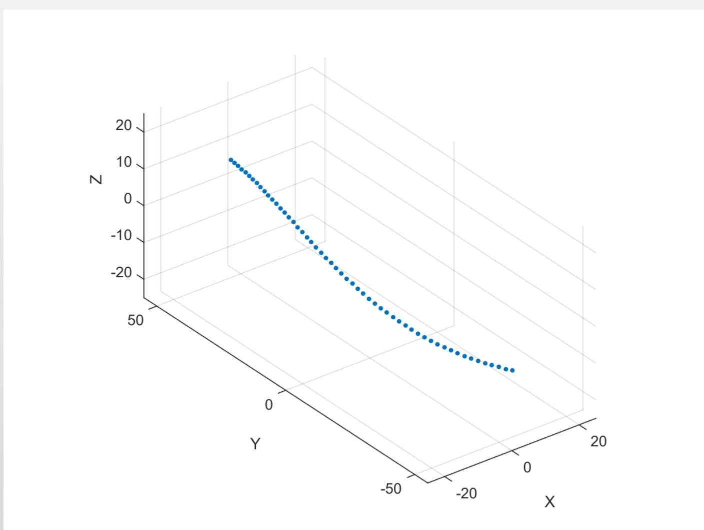
Quantify entanglement between two subsystems via their mutual information:

$$I(p:q) \equiv S_p + S_q - S_{pq}$$
$$\geq \frac{(\langle \mathcal{O}_p \mathcal{O}_q \rangle - \langle \mathcal{O}_p \rangle \langle \mathcal{O}_q \rangle)^2}{2\|\mathcal{O}_p\|^2 \|\mathcal{O}_q\|^2}$$

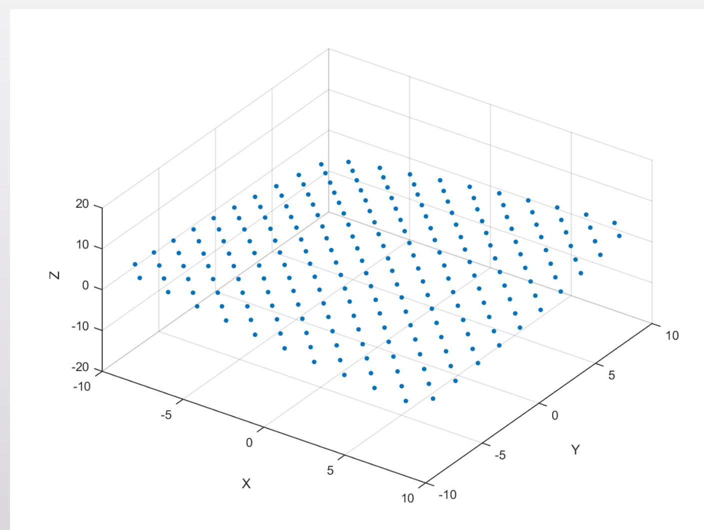
Define a distance measure in terms of decreasing mutual information between Hilbert-space factors.



Sanity check: recover simple known examples.



1-d antiferromagnetic Heisenberg chain



2-d toric code

So we have:

Geometry ⇔ Entanglement

Entanglement ⇔ Entropy

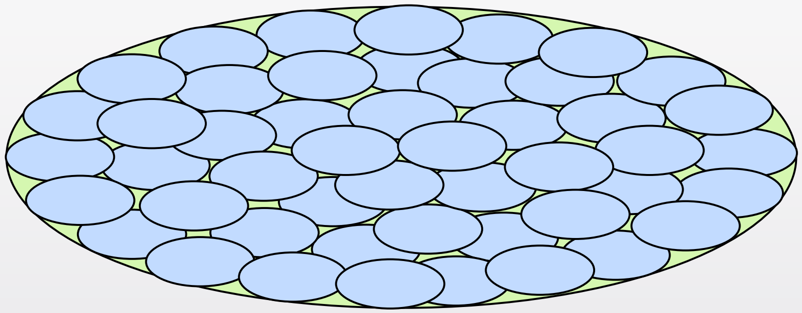
Final ingredient:

Entropy ⇔ Energy

Entropy is naturally related to energy (heat): $dS = dQ/T$

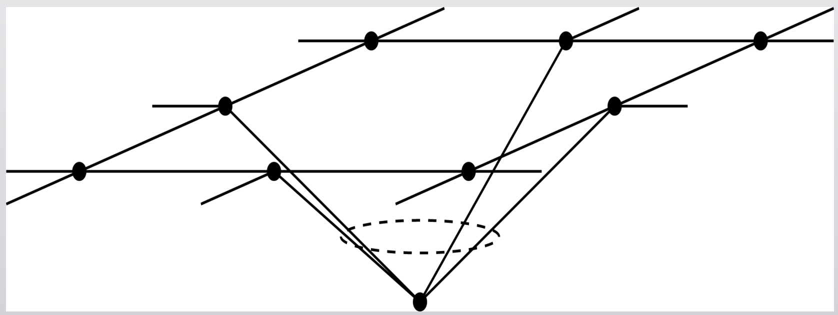
Continues to be true in quantum mechanics: change in entanglement is proportional to change in energy. "Entanglement First Law," $\delta S[\hat{\rho}] = \delta \langle \hat{K} \rangle_{\hat{\rho}}$.

Vibrating modes are naturally entangled; it takes energy to break the entanglement.



Putting it all together we're left with:

Geometry = Energy



[Faulkner et al. 2014; Jacobson 2015]

Einstein's Equation from Entanglement

(emergent space)

area $\delta\mathcal{A}$ ⟷ δS entropy

(math) ⇕   ⇕ (entanglement 1st law)

spatial curvature $\mathcal{R}$ $\delta\langle\hat{K}\rangle$ modular Hamiltonian

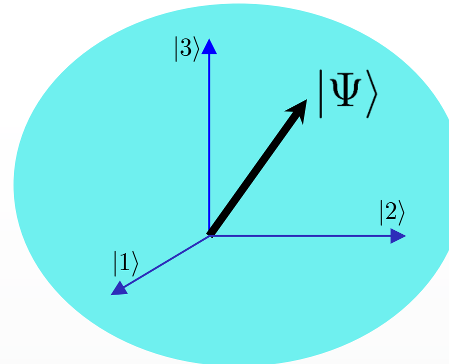
(math) ⇕   ⇕ (IR limit)

Einstein tensor G_{00} T_{00} energy density

⇕ (Lorentz invariance)

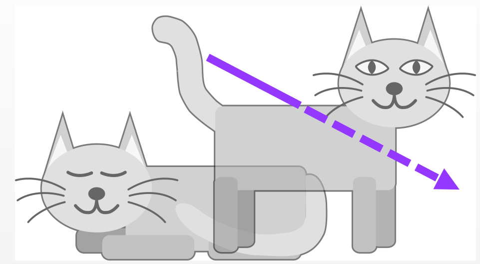
Einstein's equation $G_{\mu\nu} \propto T_{\mu\nu}$

[Cao, Carroll & Mikhalakis 2016; Cao & Carroll 2018; cf. Jacobson 1995, 2015]

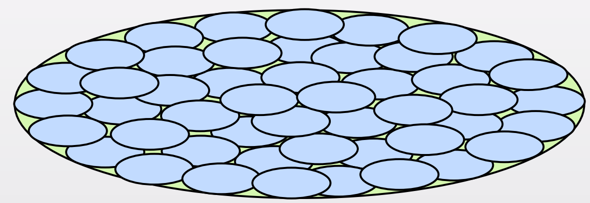


Aspirational program, in brief:

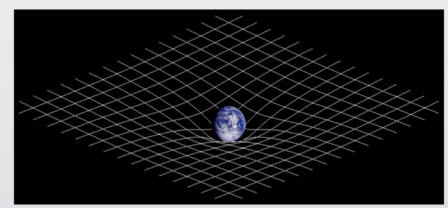
Starting with nothing but Hilbert space and a Hamiltonian,



Carve Hilbert space into system and environment by demanding classicality.



Using entanglement between different parts of the wave function to define an emergent geometry,



We are naturally led to a classical limit including Einstein's general relativity.

Many issues remain, and the program could crash and burn at any time. But perhaps we're not far from understanding the quantum origin of spacetime itself.