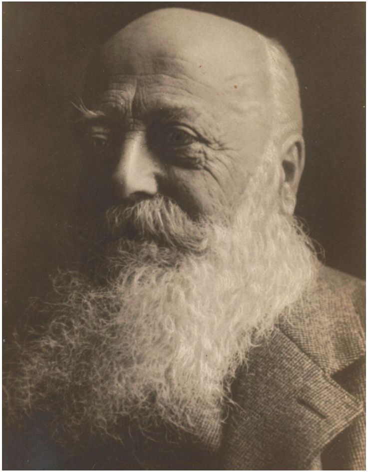
On the Maurer-Cartan Equation

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It also appears in theoretical physics.

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These ideas can be encoded together in the BV-BRST formalism (Batalin-Vilkovisky; Becchi-Rouet-Stora, Tyutin).

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One can express $s = \{S, -\}$ in terms of an anti-bracket with a generalized action S. The master equation becomes $\{S, S\} = 0$.

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Why?

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$(P^\bullet, d)$ is a *resolution* for L if there is a comparison map and $L = H^0(P)$ while $H^i(P) = 0$ for $i \neq 0$.

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Other homology groups appear when we do things to P. They are meaningful, measure deformations, obstructions etc.

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The Maurer-Cartan equation expresses that our operator has square zero (up to homology), thus it can be used to perturb homology.

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There is a deep and general framework capturing these and many other examples, *derived deformation theory*. Cf. work by Deligne, Drinfeld, Kontsevich-Soibelman, Lurie . . .

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Local systems

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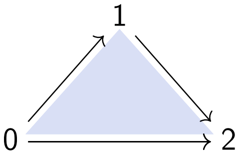

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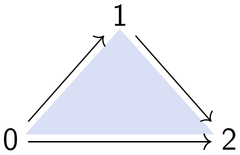

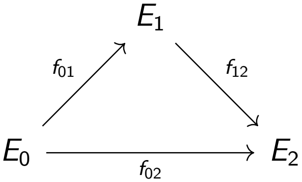

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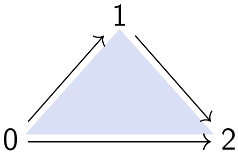

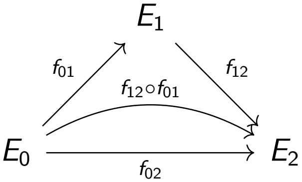

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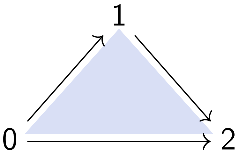

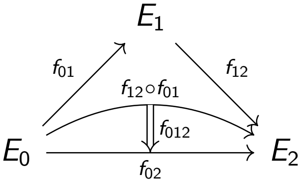

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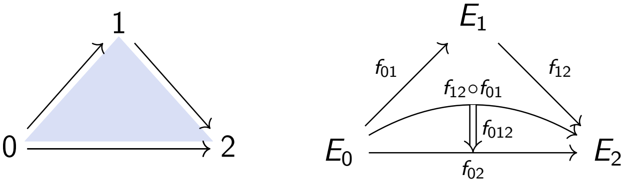

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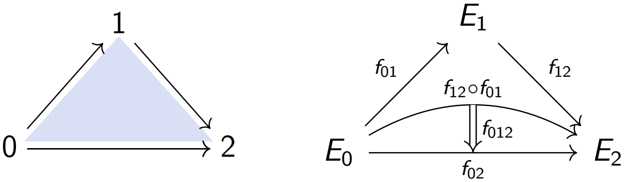

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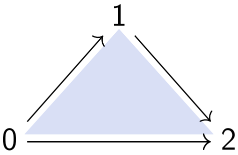

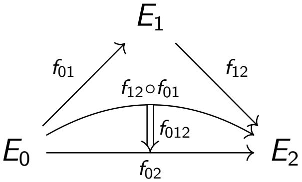

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Here f_{012} witnesses that f_{02} and $f_{12} \circ f_{01} : E_0 \to E_2$ agree in cohomology:

$$df_{012} - f_{02} + f_{12} \circ f_{01} = 0$$

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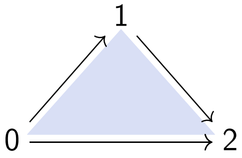
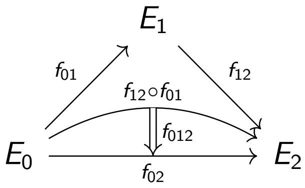
Consider a ∞-local system of differential graded vector spaces on a simplical set (simplicial complex, topological space).





For each vertex i there is a differential graded vector space E_i. For each edge or path ij there is a map $f_{ij} : E_i \to E_j$. For each 2-simplex or (topological) homotopy ijk there is a (algebraic) homotopy $f_{ijk} : E_i \to E_k[1]$.

Here f_{012} witnesses that f_{02} and $f_{12} \circ f_{01} : E_0 \to E_2$ agree in cohomology:

$$df_{012} - f_{02} + f_{12} \circ f_{01} = 0$$

There are higher relations:

$$df_{ijklm} - f_{iklm} + f_{ijlm} - f_{ijkm} + f_{jklm} \circ f_{ij} - f_{klm} \circ f_{ijk} + f_{lm} \circ f_{ijkl} = 0$$

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We remember the differential is a boundary. Indeed the boundary of the simplex induces a differential on the functions $f_{ij\ldots p}$. It sends $f_{012} \mapsto f_{02}$.

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We may also view this is arising from resolving not an individual vector space but the category of chain complexes (H).

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100 years on there is still much to be learnt about the Maurer-Cartan equation.